August 2010

Revised version, October 2010

Localization with a Surface Operator, Irregular Conformal Blocks and Open Topological String

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Abstract

Following a recent paper by Alday and Tachikawa, we compute the instanton partition function in the presence of the surface operator by the localization formula on the moduli space. For $SU(2)$ theories we find an exact agreement with CFT correlation functions with a degenerate operator insertion, which enables us to work out the decoupling limit of the superconformal theory with four flavors to asymptotically free theories at the level of differential equations for CFT correlation functions (irregular conformal blocks). We also argue that the K theory (or five dimensional) lift of these computations gives open topological string amplitudes on local Hirzebruch surface and its blow ups, which is regarded as a geometric engineering of the surface operator. By computing the amplitudes in both A and B models we collect convincing evidences of the agreement of the instanton partition function with surface operator and the partition function of open topological string.

1 Introduction

In the problem of the non-perturbative physics of four dimensional gauge theory the connection to two dimensional theory has been an useful idea. For instanton effects in the low energy effective action (F-term) of $\mathcal{N} = 2$ supersymmetric gauge theories the seminal work of Nekrasov [1] gives a combinatorial formula of the instanton partition function, which reminds us of the theory of free fermions and bosons in two dimensional conformal field theory (CFT). Last year this expectation was made quite explicit by AGT relation [2]. The holomorphic version of their proposal tells a relation of the homological (four dimensional) instanton partition function of $\mathcal{N} = 2$ (quiver) gauge theories and appropriate conformal blocks. Subsequently this correspondence was extended to incorporate loop and surface operators in four dimensional gauge theory [3] (see also [4, 5]).

In this paper we consider the instanton partition function in the presence of a surface operator and its relation to CFT correlation function with a degenerate field insertion. In a last few months there appeared several works where related ideas have been developed [6, 7, 8, 9, 10]. We note that most of them (except [7]) assume the extension of AGT relation proposed in [3] and discuss the partition function with surface operators by computing the corresponding CFT correlation functions and/or topological string amplitudes. However, as is clearly explained in [7] the computation of the instanton partition function can be made more directly by localization formula on the gauge theory side, if we consider the moduli space of instantons which involves a certain type of surface operator. In a sense this is a natural extension of the method which was used by Nekrasov to derive his formula of the instanton partition function. Based on the equivariant character formula derived by Feigin et. al. [11], we first present a few examples of direct computations of the instanton partition function with a surface operator. Precisely speaking the formula in [11] is expected to hold when the residual gauge symmetry on the surface is the maximal abelian subgroup $U(1)^N \subset U(N)$, which was called the full surface operator in [7]. But the surface operator which was argued to correspond to the degenerate operator insertion is the simple surface operator on which the gauge symmetry is reduced to $U(1) \times U(N-1) \subset U(N)$. Fortunately for the gauge group $U(2)$ these two types of the surface operator coincide. Since we rely on this coincidence, we only consider $U(2)$ gauge theories in this paper. After decoupling the diagonal $U(1)$ part, they describe $SU(2)$ theories.

The original AGT relation was proposed for the superconformal gauge theories, which are obtained by compactifying the world volume theory of $M5$ branes on an appropriate Riemann surface with punctures [12]. Recent papers on the extension of AGT relation with surface operator [6, 7, 8, 9, 10] mainly considered the superconformal case. However, the AGT relation can be generalized to asymptotically free theories [13, 14]. In this paper we will focus on $SU(2)$ theories where the number of flavors is in the region $0 \leq N_f \leq 3$. According to [3] for superconformal theories we should look at the conformal blocks with a degenerate primary operator $\Phi_{1,2}$ insertion. On the other hand in the nonconformal case we have to replace the Virasoro highest weight states with the so-called Gaiotto states [13], or an analogue of the Whittaker vector for the Virasoro algebra [15, 16]. We derive the differential equations for the one point function of $\Phi_{1,2}$ operator with respect to the Gaiotto states in a systematic manner following the appendix of [17]. In contrast to the differential equations for the usual conformal blocks, our differential equations have irregular singularities. We then obtain solutions to the differential equations which can be compared with the instanton partition function, namely those in the form of a power series in the scale parameter Λ which appears in the definition of the Gaiotto state on the CFT side. We show that they agree to the results from the localization formula on the moduli space. We emphasize that the agreement is established beyond the semi-classical limit which was argued in [3]. That is we do not have to take the limit $\epsilon_1, \epsilon_2 \to 0$ for the equality. This becomes possible, since we are able to compute an exact instanton partition function by the localization formula. On the gauge theory side the asymptotically free theories are obtained rather easily by taking the decoupling limit of $\mathcal{N} = 2$ $SU(2)$ theory with four flavors, where we take some of the masses of matter hypermultiplets into infinity and redefine the parameter Λ of instanton expansion. However, it is not straightforward to achieve the corresponding limit at the level of differential equations on the CFT side. Hence, we carefully work out the degeneration of the differential equations with irregular singularities, which describes the reduction of the number of flavors. Note that the irregular singularities appear as a consequence of the congruence of regular singularities. As a byproduct we can also see how the Gaiotto state arises from a degeneration of two Virasoro primaries.

As is expected from the idea of geometric engineering [18] the instanton partition function without surface operator is related to the topological string amplitudes [19, 20,

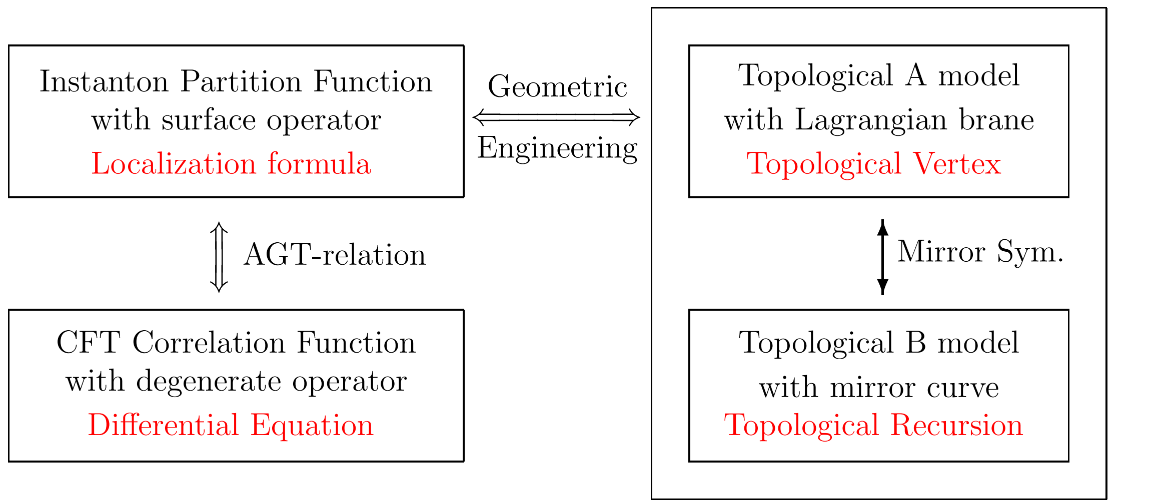

Figure 1: Correspondence among instanton counting, CFT and topological string

21, 22, 23]. Namely when the equivariant parameters (or the Ω background) (ϵ_1, ϵ_2) satisfy the self-dual condition $\epsilon_1 + \epsilon_2 = 0$, the five dimensional lift (K theory version) of the instanton partition function agrees exactly with the closed topological string partition function on the local toric Calabi-Yau manifold whose toric diagram is dictated by the geometric engineering. Since the closed topological string amplitudes compute the index of BPS states (the Gopakumar-Vafa invariants), the Nekrasov partition function in general Ω background is expected to give a refinement of the BPS state counting in topological string theory [23, 24]. As the presence of surface operators breaks half of the supersymmetry and the semi-classical part of the partition function with surface operator is identified with the twisted superpotential [3], a natural generalization of the above geometric engineering is to look at open topological strings, which has been advocated by Gukov [25]. In the second half of the paper we explore the idea of geometric engineering of the surface operator in $\mathcal{N} = 2$ gauge theories. As was proposed by Ooguri and Vafa [26] the open topological string amplitudes (open BPS invariants) give the knot and link invariants via the relation to the Chern-Simons theory with the Wilson loop operator. As the dimensional reduction to three dimensions reduces the surface operator to the loop operator, the relation to the open topological string is natural also from the view point of three dimensional Chern-Simons theory.

Pure $SU(2)$ Seiberg-Witten theory is geometrically engineered by the local Hirzebruch surface $K_{\mathbf{F}_0}$ (the total space of the canonical bundle of $\mathbf{F}_0 = \mathbf{P}^1_b \times \mathbf{P}^1_f$). The local Calabi-Yau manifold $K_{\mathbf{F}_0}$ has two moduli parameters t_b and t_f, which represent the Kähler parameters of the base $\mathbf{P}^1_b$ and the fiber $\mathbf{P}^1_f$, respectively. The parameter of the instanton expansion (the dynamical mass scale) Λ and the vacuum expectation value a of the scalar field in the prepotential of $\mathcal{N} = 2$ theory are related to these moduli parameters by $(\beta\Lambda)^4 \sim e^{-t_b}$ and $2\beta a \sim t_f$ with β being a scale parameter of length. By blowing up at N_f points in toric geometry we can add N_f matter hypermultiplets in the fundamental representation. The corresponding geometry is described by the local toric del Pezzo surfaces. It has been argued that the (simple) surface operator is geometrically engineered by a toric Lagrangian brane inserted on the inner edge of the toric diagram which corresponds to the base $\mathbf{P}^1_b$ of the surface [8]. We compute topological open string amplitudes on this local toric Calabi-Yau geometry with a brane in both A and B model perspectives. The disk amplitude, which corresponds to the superpotential, is most easily computed by the B model approach, since it is naturally related to the period integral. We first use the Seiberg-Witten curve which can be associated to the semi-classical limit of the expectation value of the energy-momentum tensor on the CFT side. We also make computations based on the mirror curve of the local Calabi-Yau geometry. In both cases we can show an agreement with CFT correlation functions with a degenerate field insertion. We employ the method of remodeling [27, 28] in our B model computations. One of the advantages of this method is that we can easily increase the number of holes (boundaries) of the world sheet by the topological recursion relation coming from the matrix model [29]. Motivated by a recent suggestion in [6], we also compare annulus amplitude and three hole amplitude with CFT correlation functions with multiple insertion of $\Phi_{1,2}$ operator. We again find a matching of both computations as far as the comparison is possible.

For the A model computation we use the powerful method of the topological vertex [30]. We first look at the decoupling limit of four dimensional gauge theory from the two dimensional theory on the surface. As argued in [8] in this limit the partition function is reduced to the generating function of the vortex counting. We show that the vortex counting in [8] can be successfully recovered from the localization formula on the affine Laumon space. From the viewpoint of four dimensional theory only the sector of vanishing

instanton number survives in this decoupling limit. Thus the next task is to examine the sector of instanton number one. The corresponding part of topological string amplitudes is the first order term in the Kähler moduli parameter t_b of the base $\mathbf{P}^1_b$. We check that in this order the open topological string amplitude on the local Hirzebruch surface exactly agrees with the instanton partition function with a surface operator modulo a partial shift of the Kähler moduli t_f of the fiber $\mathbf{P}^1_f$ by the parameter of the Ω background. We conjecture this shift becomes trivial in the limit $\epsilon_2 \to 0$, while keeping ϵ_1 finite. Note that such a limit appears in the recent proposal of a quantization of the integrable system associated with the Seiberg-Witten geometry [31] (see also [32, 33, 34] and a more recent discussion [9]). It is desirable to understand the origin of the shift as an effect of the presence of the surface operator, or the insertion of $\Phi_{1,2}$ operator to CFT correlation functions. The computations of topological string amplitudes in this paper are subjected to the condition $\epsilon_1 + \epsilon_2 = 0$. In the A model the amplitudes in general Ω background (ϵ_1, ϵ_2) can be computed by the refined topological vertex [35, 36], but the computation gets rather involved. The validity of the above conjecture should be checked by computing the refined topological string amplitudes. We leave these issues to future works.

The paper is organized as follows: In the next section we introduce the instanton partition function with surface operator and review some of mathematical background for the relevant moduli space. In section 3, following the prescription described in [7], we compute the instanton partition function for pure $SU(2)$ theory as a basic example of the application of the localization formula. We also consider $N_f = 4$ theory from which asymptotically free theories with $N_f < 3$ are obtained by the decoupling limit. The instanton partition functions computed by the localization formula are compared with the corresponding CFT correlation functions in section 4. We have to multiply appropriate overall factors for the matching. The origin of the factor is clarified in section 5, where the degeneration of the differential equations for irregular conformal blocks is derived from the consistency with the decoupling of the hypermultiplets on the gauge theory side. The latter half of the paper is devoted to the geometric engineering of the half-BPS surface operator in $\mathcal{N} = 2$ theories. In sections 6 and 7 we take the B model approach based on the topological recursion relation. In section 8 we compute the A model amplitude by the method of the topological vertex. Basic formulas and some of

technical details are collected in Appendices.

2 Instanton partition function with surface operator

In [3] the semi-classical matching of the instanton partition function in the presence of a surface operator and the conformal block with the insertion of a degenerate field was pointed out. To establish a full agreement beyond the semi-classical limit we have to set up an appropriate framework of the instanton counting that incorporates the surface operator. In this section we review a few mathematical backgrounds following [3, 7] and try to make the definition of the partition function as clear as possible, since a proper definition of the moduli space is required to justify the computation of the partition function by the equivariant localization.

Recall that one of the ways to define the surface operator is to prescribe a singular behavior of the gauge field [37] (see also [38, 39] for the surface operators in $\mathcal{N} = 2$ theories and [40, 41] for more mathematical formulation). Let us consider a gauge field A_μ on $\mathbb{R}^4 \simeq \mathbb{C}^2$ with complex coordinates (z_1, z_2) and assume that there is a surface operator at $z_2 = 0$ which fills the z_1-plane. If θ is the angular coordinate of the transverse plane (the z_2-plane) to the surface operator, the gauge field diverges as

$$A_\mu dx^\mu \sim \mathrm{diag}\ (\alpha_1, \alpha_2, \cdots, \alpha_N)\ id\theta, \tag{2.1}$$

near the support $\mathcal{S} := \{(z_1, z_2)|z_2 = 0\}$ of the surface operator. Note that the data $(\alpha_1, \alpha_2, \cdots, \alpha_N)$ which characterize the surface operator give an element of the Lie algebra of the maximal Abelian subgroup $U(1)^N$ of the gauge group $G = U(N)$. Then we can associate a Young diagram with N boxes (a partition of N): $N = N_1 + N_2 + \cdots + N_s$, if diag $(\alpha_1, \alpha_2, \cdots, \alpha_N)$ commutes with $\mathbb{L} := U(N_1) \times U(N_2) \times \cdots \times U(N_s)$. From the viewpoint of the principal G-bundle this means the structure group is reduced to a Levi subgroup $\mathbb{L} \subset G$ on the surface. The subgroup $\mathbb{L}$ is identified with the Levi part of a parabolic subgroup P of the complexified Lie group $G_{\mathbb{C}} = GL(N)$. By a gauge transformation we may assume $\alpha_i \geq \alpha_{i+1}$. When α_i are the most generic, the commutant is $U(1)^N$ and the corresponding parabolic subgroup becomes minimal one, namely the Borel subgroup B of $GL(N)$. The corresponding surface operator is called full surface operator in [7]. Note that since we have fixed the ordering $\alpha_1 > \alpha_2 > \cdots > \alpha_N$, the Weyl invariance is lost. We will see its effect on the instanton partition function in the next

section. Following the terminology used in the context of $\mathcal{N}=4$ gauge theory [37], we call the instantons with the singular behavior (2.1) "ramified" instantons[1]. The "ramified" instantons are anti-self-dual connections on $\mathbb{R}^4 \setminus \mathcal{S}$ and their topological indices are the instanton number k and the monopole number

$$\mathfrak{m} := \frac{1}{2\pi}\int_{\mathcal{S}} F \in \Lambda_{\mathbb{L}} \simeq H_2(G/\mathbb{L}, \mathbb{Z}). \tag{2.2}$$

For the full surface operator we can see the origin of the monopole number as follows: Since the gauge group on the surface is reduced to $\mathbb{L}=U(1)^N$ in this case, we have N abelian gauge fields or line bundles $L_1, L_2, \cdots L_N$ on the surface. Hence the "ramified" instanton has N monopole numbers

$$\mathfrak{m}_i := \frac{1}{2\pi}\int_{\mathcal{S}} F_i = \int_{\mathcal{S}} c_1(L_i), \qquad i=1,2,\cdots N. \tag{2.3}$$

The generating function of the instanton counting with surface operator is defined by

$$Z^{(S)}_{\text{inst}} = \sum_{k=0}^{\infty}\sum_{\mathfrak{m}\in\Lambda_{\mathbb{L}}} q^k z^{\mathfrak{m}} \int_{\mathcal{M}_{N,(k,\mathfrak{m})}} \mathbf{1}, \tag{2.4}$$

where q is a parameter of instanton expansion and $\mathcal{M}_{N,(k,\mathfrak{m})}$ is the moduli space of "ramified" $U(N)$ instantons with instanton number k and the monopole number $\mathfrak{m}$. If the theory is superconformal, we can relate the expansion parameter to the gauge coupling τ by $q=e^{2\pi i\tau}$. For asymptotically free theories it is replaced with the parameter of dynamical scale Λ with appropriate mass dimension. If we put the expansion parameter z associated with the monopole number to $z=e^{2\pi i t}$, then the parameter t has the following meaning. As was argued by Gukov and Witten [37] in $\mathcal{N}=4$ gauge theories, the surface operator may be described by a coupling of four dimensional gauge theory to a two dimensional sigma model on the surface $\mathcal{S}$ with the target $G/\mathbb{L} \simeq G_{\mathbb{C}}/P$. Then the parameter t is identified with the complexified Kähler moduli of the flag manifold $G_{\mathbb{C}}/P$. From the view point of the sigma model the monopole number $\mathfrak{m}$ measures the degrees of the map $\Phi : \mathcal{S} \to G/\mathbb{L}$. For example, when $\mathbb{L}=U(1)\times SU(N-1) \subset U(N)$, the target space is the projective space $\mathbf{CP}^{N-1}$ and t is the complexified Kähler moduli of the projective space, which is one dimensional. In this case the monopole number is a single integer and the corresponding surface operator is called simple [7].

[1] The name "ramified" comes from the fact that the ramification in the (geometric) Langlands problem is related to the presence of a surface operator, or a codimension two singularity in gauge theory.

As was discussed in [7] it is convenient to combine the instanton number and monopole numbers to define a vector $\vec{k} = (k_1, k_2, \cdots, k_N)$ as follows[2]:

$$k_1 = k, \quad k_{i+1} - k_i = \mathfrak{m}_i. \tag{2.5}$$

The moduli space $\mathcal{M}_{N,\vec{k}}$ of the "ramified" instantons with the topological number $\vec{k}$ has real dimension $4(k_1 + k_2 + \cdots + k_N)$. Since we integrate $\mathbf{1}$ over the moduli space in (2.4), we may expect it computes the volume of $\mathcal{M}_{N,\vec{k}}$. However, the moduli space is highly singular and "non-compact ". Hence we have to regularize the integral. To overcome the problem we can employ the strategy that was used to derive the Nekrasov partition function. We consider a natural toric action of $\mathbb{T}$ on the moduli space and the integral is regularized as the equivariant integral, or the push-forward to the equivariant cohomology of a point $H_{\mathbb{T}}(\mathrm{pt})$. In the next section we will compute the equivariant integral by using the localization formula. But the use of the localization theorem is mathematically justified only when the moduli space is smooth. However, $\mathcal{M}_{N,\vec{k}}$ suffers from various types of singularities, which keeps us from applying the localization formula. A standard method to handle such a problem is to consider torsion free sheaves with an appropriate stability condition; see [7] and literatures in mathematics cited therein. The use of torsion free sheaves for the instanton counting without surface operators is clearly explained in [42, 43]. It is shown that torsion free sheaves are also useful for constructing a Uhlenbeck space for the instantons with a parabolic structure [44]. For general gauge group G the existence of a smooth moduli space is still open problem, even if we shift the construction of a smooth moduli space to the problem of torsion free sheaves. Fortunately for $G = U(N)$ a resolution of singularities $\widetilde{\mathcal{M}}_{N,\vec{k}} \to \mathcal{M}_{N,\vec{k}}$ (called small resolution in mathematics) is successfully constructed[3]. The smooth moduli space $\widetilde{\mathcal{M}}_{N,\vec{k}}$ can be regarded as an affine version of the Laumon space and called affine Laumon space in mathematics [45, 46]. According to the description in [7] it consists of the equivalence classes of the following data up to gauge transformations:

- stable rank N torsion free sheaves on $\mathbf{P}^1 \times \mathbf{P}^1$ with a given topological number $\vec{k}$,
- a fixed framing at infinity $\{z_1 = \infty\} \cup \{z_2 = \infty\}$,

[2] In [15, 16] it was pointed out that it is natural to combine k with $\mathfrak{m}_i$ from the viewpoint of the affine Lie algebra.

[3] We would like to thank K. Nagao for explaining this fact.

- a reduction of the gauge group $GL(N)$ to a parabolic subgroup P on the surface $\{z_2 = 0\}$, which is called a parabolic structure.

It is remarkable that in the definition of the affine Laumon space $\widetilde{\mathcal{M}}_{N,\vec{k}}$, a "compactification" of $\mathbb{C}^2$ is given not by $\mathbf{P}^2$ but by $\mathbf{P}^1 \times \mathbf{P}^1$. The standard toric action $(z_1, z_2) \to (e^{i\epsilon_1} \cdot z_1, e^{i\epsilon_2} \cdot z_2)$ on $\mathbb{C}^2$ survives after this "compactification". Thus we can consider the fixed point of the toric action of $\mathbb{T} := U(1)^2 \times U(1)^N \subset SO(4) \times U(N)$ on the moduli space of "ramified" instantons, which is familiar in the computation of the Nekrasov partition function. In [11] it was shown that the fixed point is isolated and labeled by an N-tuple of Young diagrams $\vec{\lambda} = (\lambda_1, \lambda_2, \cdots \lambda_N)$. However, we should warn that the manner how these Young diagrams appear is rather different from the case of the standard instanton where the moduli space is constructed by ADHM data. In fact the constraints imposed on the N-tuple of Young diagrams $\vec{\lambda}$ are

$$k_i = k_i(\vec{\lambda}) := \sum_{j \geq 0} \lambda_{i+j,j+1}, \tag{2.6}$$

where $\lambda_{i,j}$ is the length of the j-th row of the Young diagram λ_i and we define $\lambda_{i,j}$ for $i > N$ by requiring $\lambda_{i+N} \equiv \lambda_i$. Thus the fixed points on the moduli space $\widetilde{\mathcal{M}}_{N,\vec{k}}$ are in one to one correspondence with $\vec{\lambda}$ that satisfies the condition (2.6). Because the affine Laumon space is smooth, we can consider the tangent space at each fixed point with complex dimension $2(k_1 + k_2 + \cdots + k_N)$. At fixed points the toric action induces a structure of $U(1)^2 \times U(1)^N$ module on the tangent space. In [11] the structure of this module was determined and a formula of equivariant character was provided, which allows us to compute the instanton partition function (2.4) by the localization theorem.

It is amusing that a closely related moduli space was already appeared in a proof of the Nekrasov conjecture from the viewpoint of integrable system and the representation theory of the affine Lie algebra [15, 16], where the moduli space of the instantons with parabolic structure was introduced. In [15, 16] the Uhlenbeck compactification of the moduli space [44] and a sophisticated theory of the intersection cohomology were used to compute the equivariant integral. On the other hand the affine Laumon space provides a semi-flat resolution of singularities and we can apply the standard theory of the equivariant cohomology and the localization theorem to compute our partition function (2.4).

3 Equivariant localization on affine Laumon space

In this section, following the method of computation in [7], we work out a few examples of the instanton partition function in the presence of the surface operator by localization formula. As was discussed in the last section if the gauge group is $U(2)$, the fixed points are isolated and labeled by a pair of Young diagrams The measure of the localization formula at each fixed point is obtained from explicit computations of the equivariant character of toric action on the affine Laumon space. Fortunately we have a formula of the equivariant character derived in [11], which is given in Appendix A (see also eq. (3.10) in [7]).

3.1 Pure Yang-Mills theory

Let us assume that all the fields in the theory are in the adjoint representation. The so called $\mathcal{N} = 2^*$ theory with a massive adjoint matter, which is a deformation of $\mathcal{N} = 4$ conformal theory is a typical example. Pure Yang-Mills theory, which can be obtained by decoupling the adjoint matter of $\mathcal{N} = 2^*$ theory, is an example of asymptotically free theories. In this case we only need the diagonal component of the equivariant character provided in Appendix A, where we set $a = b$ and $\lambda = \mu$. The fixed points of the toric action are labeled by a pair of partitions $\vec{\lambda} := (\lambda_1, \lambda_2)$ and $\lambda_{m,n} = \lambda_{m+2,n}$ denotes the n-th component of the partition λ_m. The vacuum expectation values of the scalar fields a_k are also defined with $a_{k+2} \equiv a_k$. At each fixed point $\vec{\lambda}$ the formula in Appendix A gives many terms in general. But after several cancellations the final result should be a sum of $2|\vec{\lambda}|$ monomials with $|\vec{\lambda}| = \sum_{n=1}^{\infty}(\lambda_{1,n} + \lambda_{2,n})$:

$$\mathrm{ch}(a_k, \epsilon_1, \epsilon_2) := \mathrm{Tr}_{\mathrm{Ext}(\vec{\lambda},\vec{\lambda})} g = \sum_{i=1}^{2|\vec{\lambda}|} e^{s_i}, \tag{3.1}$$

where each power s_i is a linear combination of ϵ_1, ϵ_2 and a_k.

By the localization theorem the instanton partition function with a (full) surface operator is computed as follows [7]:

$$Z^{(S)}(x, y; \epsilon_1, \epsilon_2, a, m) = \sum_{\vec{\lambda}} x^{k_1(\vec{\lambda})} y^{k_2(\vec{\lambda})} \frac{n_{\mathrm{matter}}(\vec{\lambda}; a, m)}{n_{\mathrm{gauge}}(\vec{\lambda}, a)}. \tag{3.2}$$

The equivariant character (3.1) gives $n_{\mathrm{matter}}(\vec{\lambda}; a, m) := n_{\mathrm{adj}}(\vec{\lambda}; a, m)$ for the adjoint hypermultiplet with mass m and $n_{\mathrm{gauge}}(\vec{\lambda}, a) := n_{\mathrm{adj}}(\vec{\lambda}; a, 0)$ for the vector multiplet. For cohomology version we have

$$n_{\mathrm{adj}}(\vec{\lambda}; a, m) = \prod_{i=1}^{2|\vec{\lambda}|}(s_i - m), \tag{3.3}$$

while for K-theory version it is

$$n_{\mathrm{adj}}(\vec{\lambda}; a, m) = \prod_{i=1}^{2|\vec{\lambda}|} 2\sinh((s_i - m)/2). \tag{3.4}$$

In (3.2) x, y are (formal) expansion parameters and topological numbers are defined by

$$k_1(\vec{\lambda}) := \sum_{n\geq 1}\lambda_{1,2n-1} + \sum_{n\geq 1}\lambda_{2,2n}, \qquad k_2(\vec{\lambda}) := \sum_{n\geq 1}\lambda_{1,2n} + \sum_{n\geq 1}\lambda_{2,2n-1}. \tag{3.5}$$

We see that $k_1 + k_2 = |\vec{\lambda}|$. The relation to the instanton number k and the monopole charge $\mathfrak{m}$ on the surface is given by

$$k = k_1, \qquad \mathfrak{m} = k_2 - k_1. \tag{3.6}$$

Note that we have both positive and negative monopole charges.

In view of the comparison with CFT correlation functions let us look at the cohomology version:

$$Z^{(S)}_{\mathcal{N}=2^*}(x, y; \epsilon_1, \epsilon_2, a, m) = \sum_{\vec{\lambda}} x^{k_1(\vec{\lambda})} y^{k_2(\vec{\lambda})} \prod_{i=1}^{2|\vec{\lambda}|} \frac{s_i - m}{s_i}. \tag{3.7}$$

This is the instanton partition function for the mass deformed $\mathcal{N} = 4$ theory. In the massless limit it just counts the number of fixed points with weight $x^{k_1(\vec{\lambda})} y^{k_2(\vec{\lambda})}$. In the decoupling limit ($m \to \infty$), by renormalizing the parameters x, y by m^2, we have

$$Z^{(S)}_{\mathrm{N_f}=0}(\Lambda_i; \epsilon_1, \epsilon_2, a) = \sum_{\vec{\lambda}} \Lambda_1^{k_1(\vec{\lambda})} \Lambda_2^{k_2(\vec{\lambda})} \prod_{i=1}^{2|\vec{\lambda}|} \frac{1}{s_i}, \tag{3.8}$$

where $\Lambda_1 = m^2 x$ and $\Lambda_2 = m^2 y$. The condition of vanishing monopole charge is $k := k_1(\vec{\lambda}) = k_2(\vec{\lambda})$ and in this case $|\vec{\lambda}| = 2k$. Restricting to this sector the partition function becomes

$$Z^{(\mathfrak{m}=0)}_{\mathrm{N_f}=0}(\Lambda_i; \epsilon_1, \epsilon_2, a) = \sum_{k=k_1(\vec{\lambda})=k_2(\vec{\lambda})} (\Lambda_1\Lambda_2)^k \prod_{i=1}^{4k} \frac{1}{s_i}. \tag{3.9}$$

From the formula in Appendix A we have computed the characters $\mathrm{ch}(a_k, \epsilon_1, \epsilon_2)$ for lower instanton numbers. We can see that in general the character at (λ_2, λ_1) is obtained from that at (λ_1, λ_2) by the transformation $(a_1 - a_2) \to (a_2 - a_1) - \epsilon_2$ and $(a_2 - a_1) \to (a_1 - a_2) + \epsilon_2$. Our computation gives the following partition function for pure gauge theory:

$$
\begin{aligned}
&Z^{(S)}_{\mathrm{N_f}=0}(\Lambda_i; \epsilon_1, \epsilon_2, a) \\
&= 1 + \frac{1}{\epsilon_1(-2a+\epsilon_1)}\Lambda_1 + \frac{1}{\epsilon_1(2a+\epsilon_1+\epsilon_2)}\Lambda_2 \\
&\quad + \frac{1}{2{\epsilon_1}^2(-2a+\epsilon_1)(-2a+2\epsilon_1)}\Lambda_1^2 + \frac{1}{2{\epsilon_1}^2(2a+\epsilon_1+\epsilon_2)(2a+2\epsilon_1+\epsilon_2)}\Lambda_2^2 \\
&\quad + \left(\frac{1}{\epsilon_1\epsilon_2(2a)(-2a+\epsilon_1)} + \frac{1}{{\epsilon_1}^2(-2a)(2a+\epsilon_2)} + \frac{1}{\epsilon_1\epsilon_2(2a+\epsilon_1+\epsilon_2)(-2a-\epsilon_2)}\right)\Lambda_1\Lambda_2 \\
&\quad + \frac{1}{6\epsilon_1^3(-2a+\epsilon_1)(-2a+2\epsilon_1)(-2a+3\epsilon_1)}\Lambda_1^3 \\
&\quad + \left(\frac{1}{{\epsilon_1}^2(-\epsilon_1+\epsilon_2)(2a)(-2a+\epsilon_1)(-2a+2\epsilon_1)} + \frac{1}{\epsilon_1\epsilon_2(\epsilon_1-\epsilon_2)(2a)(-2a+\epsilon_1)(-2a+\epsilon_1+\epsilon_2)}\right. \\
&\quad \left. + \frac{1}{2{\epsilon_1}^3(2a-\epsilon_1+\epsilon_2)(-2a)(-2a+\epsilon_1)} + \frac{1}{{\epsilon_1}^2\epsilon_2(-2a)(-2a+\epsilon_1-\epsilon_2)(2a+\epsilon_1+\epsilon_2)}\right)\Lambda_1^2\Lambda_2 \\
&\quad + \left(\frac{1}{{\epsilon_1}^2\epsilon_2(-2a+\epsilon_1)(2a+\epsilon_1)(2a+\epsilon_2)} + \frac{1}{2{\epsilon_1}^3(-2a-\epsilon_1)(2a+\epsilon_2)(2a+\epsilon_1+\epsilon_2)}\right. \\
&\quad + \frac{1}{{\epsilon_1}^2(-\epsilon_1+\epsilon_2)(-2a-\epsilon_2)(2a+\epsilon_1+\epsilon_2)(2a+2\epsilon_1+\epsilon_2)} \\
&\quad \left. + \frac{1}{\epsilon_1\epsilon_2(\epsilon_1-\epsilon_2)(-2a-\epsilon_2)(2a+\epsilon_1+\epsilon_2)(2a+\epsilon_1+2\epsilon_2)}\right)\Lambda_1\Lambda_2^2 \\
&\quad + \frac{1}{6\epsilon_1^3(2a+\epsilon_1+\epsilon_2)(2a+2\epsilon_1+\epsilon_2)(2a+3\epsilon_1+\epsilon_2)}\Lambda_2^3 + O(\Lambda_i^4),
\end{aligned}
\tag{3.10}
$$

where we have set $a := a_1 = -a_2$. As we will see in the next section, up to this order the partition function $Z^{(S)}_{\mathrm{N_f}=0}(\Lambda_i; \epsilon_1, \epsilon_2, a)$ completely agrees to the result which is obtained from the differential equation for CFT one point function with $\Phi_{1,2}$ insertion. This means that $Z^{(S)}_{\mathrm{N_f}=0}(\Lambda_i; \epsilon_1, \epsilon_2, a)$ satisfies the differential equation in the Appendix of [17], after the substitution $\Lambda_1 = -z^{-1}\Lambda^2, \Lambda_2 = -z\Lambda^2$, where z is the position of the degenerate field insertion.

The free energy is defined by

$$
F^{(S)}_{\mathrm{N_f}=0}(\Lambda_i; \epsilon_1, \epsilon_2, a) = \log Z^{(S)}_{\mathrm{N_f}=0}(\Lambda_i; \epsilon_1, \epsilon_2, a). \tag{3.11}
$$

Using $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$, we find

$$
\begin{aligned}
&F^{(S)}_{\mathrm{N_f}=0}(\Lambda_i;\epsilon_1,\epsilon_2,a) \\
&= \frac{1}{\epsilon_1(-2a+\epsilon_1)}\Lambda_1 + \frac{1}{\epsilon_1(2a+\epsilon_1+\epsilon_2)}\Lambda_2 - \frac{1}{2\epsilon_1(-2a+\epsilon_1)^2(-2a+2\epsilon_1)}\Lambda_1^2 \\
&\quad - \frac{1}{2\epsilon_1(2a+\epsilon_1+\epsilon_2)^2(2a+2\epsilon_1+\epsilon_2)}\Lambda_2^2 - \frac{2}{\epsilon_1\epsilon_2(2a+\epsilon_1+\epsilon_2)(2a-\epsilon_1)}\Lambda_1\Lambda_2 \\
&\quad - \frac{2}{3\epsilon_1(2a-\epsilon_1)^3(2a-2\epsilon_1)(2a-3\epsilon_1)}\Lambda_1^3 + \frac{2}{3\epsilon_1(2a+\epsilon_1+\epsilon_2)^3(2a+2\epsilon_1+\epsilon_2)(2a+3\epsilon_1+\epsilon_2)}\Lambda_2^3 \\
&\quad - \frac{2}{\epsilon_1(2a-\epsilon_1)^2(2a-2\epsilon_1)(2a-\epsilon_1-\epsilon_2)(2a+\epsilon_1+\epsilon_2)}\Lambda_1^2\Lambda_2 \\
&\quad + \frac{2}{\epsilon_1(2a+\epsilon_1+\epsilon_2)^2(2a-\epsilon_1)(2a+2\epsilon_1+\epsilon_2)(2a+\epsilon_1+2\epsilon_2)}\Lambda_1\Lambda_2^2 + O(\Lambda_i^4).
\end{aligned}
\tag{3.12}
$$

Note that the higher pole of ${\epsilon_1}^{-2}$ disappears in the free energy. The above expressions are not invariant under the Weyl group action $a \to -a$ of $SU(2)$. However, by the shift $2\tilde{a} := 2a + \frac{\epsilon_2}{2}$, we may recover the invariance under $\tilde{a} \to -\tilde{a}$.

In the free energy (3.12) the pole structure of the terms with non-vanishing monopole number ($\Lambda_1^n\Lambda_2^m$, $(n \neq m)$) is ϵ_1^{-1}, while that of zero monopole part is $(\epsilon_1\epsilon_2)^{-1}$. Thus it is natural to compare the zero monopole number terms of the free energy with the Nekrasov partition function. Up to three instantons we obtain

$$
\begin{aligned}
F^{(\mathfrak{m}=0)}_{\mathrm{N_f}=0}(\Lambda_i;\epsilon_1,\epsilon_2,a) &= \frac{2\Lambda_1\Lambda_2}{\epsilon_1\epsilon_2 D_1(a)D_1(-a-\epsilon_2/2)} - \frac{N_2\Lambda_1^2\Lambda_2^2}{\epsilon_1\epsilon_2 D_2(a)D_2(-a-\epsilon_2/2)} \\
&\quad + \frac{16}{3}\frac{N_3\Lambda_1^3\Lambda_2^3}{\epsilon_1\epsilon_2 D_3(a)D_3(-a-\epsilon_2/2)} + O(\Lambda_1^4\Lambda_2^4)
\end{aligned}
\tag{3.13}
$$

with

$$
\begin{aligned}
D_1(a) &:= (2a+\epsilon_1+\epsilon_2), \\
D_2(a) &:= (2a+2\epsilon_1+\epsilon_2)(2a+\epsilon_1+\epsilon_2)^2(2a+\epsilon_1+2\epsilon_2), \\
D_3(a) &:= (2a+3\epsilon_1+\epsilon_2)(2a+2\epsilon_1+\epsilon_2)(2a+\epsilon_1+\epsilon_2)^3(2a+\epsilon_1+2\epsilon_2)(2a+\epsilon_1+3\epsilon_2), \\
N_2 &:= 20a^2 + 10\epsilon_2 a + (7\epsilon_1^2 + 11\epsilon_1\epsilon_2 + 2\epsilon_2^2), \\
N_3 &:= 144a^4 + 144\epsilon_2 a^3 + (232\epsilon_1^2 + 416\epsilon_1\epsilon_2 + 88\epsilon_2^2)a^2 + (116\epsilon_1^2 + 208\epsilon_1\epsilon_2 + 26\epsilon_2^2)\epsilon_2 a \\
&\quad + (29\epsilon_1^4 + 116\epsilon_1^3\epsilon_2 + 118\epsilon_1^2\epsilon_2^2 + 13\epsilon_1\epsilon_2^3 + 3\epsilon_2^4).
\end{aligned}
\tag{3.14}
$$

On the other hand the free energy of the Nekrasov partition function is

$$
F^{(\mathrm{Nek})}_{\mathrm{N_f}=0}(\Lambda;\epsilon_1,\epsilon_2,a) = \frac{2\Lambda^4}{\epsilon_1\epsilon_2 D_1(a)D_1(-a)} - \frac{N_2^{(\mathrm{Nek})}\Lambda^8}{\epsilon_1\epsilon_2 D_2(a)D_2(-a)} + \frac{16}{3}\frac{N_3^{(\mathrm{Nek})}\Lambda^{12}}{\epsilon_1\epsilon_2 D_3(a)D_3(-a)} + O(\Lambda^{16})
\tag{3.15}
$$

with

$$
\begin{aligned}
N_2^{(\mathrm{Nek})} &:= 20a^2 + (7\epsilon_1^2 + 16\epsilon_1\epsilon_2 + 7\epsilon_2^2), \\
N_3^{(\mathrm{Nek})} &:= 144a^4 + (232\epsilon_1^2 + 568\epsilon_1\epsilon_2 + 232\epsilon_2^2)a^2 \\
&\quad + (29\epsilon_1^4 + 154\epsilon_1^3\epsilon_2 + 258\epsilon_1^2\epsilon_2^2 + 154\epsilon_2^3\epsilon_1 + 29\epsilon_2^4).
\end{aligned} \tag{3.16}
$$

The free energy of the Nekrasov partition function is symmetric under both $a \to -a$ (the $SU(2)$ Weyl invariance) and $\epsilon_1 \leftrightarrow \epsilon_2$. However, the existence of the surface breaks these symmetries, even in the vanishing monopole sector. One may argue the origin of this discrepancy from the view point of CFT correlation function with $\Phi_{1,2}$ operator insertion. The comparison of (3.13) and (3.15) suggests a simple rule of translation between the denominators. We will encounter a similar rule in the computation of open topological string amplitudes by the topological vertex. It is very curious that up to three instantons both the free energies give the same result in the limit $\epsilon_2 \to 0$. Thus we conjecture that

$$
\lim_{\epsilon_2 \to 0} \epsilon_1\epsilon_2 \; F^{(\mathfrak{m}=0)}_{\mathrm{N_f}=0}(\Lambda_i; \epsilon_1, \epsilon_2, a) = \lim_{\epsilon_2 \to 0} \epsilon_1\epsilon_2 \; F^{(\mathrm{Nek})}_{\mathrm{N_f}=0}(\Lambda; \epsilon_1, \epsilon_2, a). \tag{3.17}
$$

with $\Lambda^4 = \Lambda_1\Lambda_2$. If we assume a complete agreement of the instanton partition function with a surface operator and the CFT correlation function with a degenerate field insertion, which we confirm in lower orders in the instanton expansion, the conjecture follows from theorem 1.6 in [16]. This is because the differential equation for the CFT correlation function with a degenerate field insertion coincides with the one derived by Braverman and Etingof [17].

3.2 $N_f = 4$ theory (superconformal case)

The equivariant character $\mathrm{Tr}_{\mathrm{Ext}(\vec{\lambda},\vec{\mu})}[\mathrm{diag.}(\epsilon_1, \epsilon_2; \vec{a}, \vec{b})]$ at a fixed point of the toric action on the affine $SU(2)$ Laumon space is given in Appendix A. After the summation over all the contributions, the equivariant character $\mathrm{Tr}_{\mathrm{Ext}(\vec{\lambda},\vec{\mu})}[\mathrm{diag.}(\epsilon_1, \epsilon_2; \vec{a}, \vec{b})]$ is expressed as a sum of $|\vec{\lambda}| + |\vec{\mu}|$ monomials:

$$
\mathrm{Tr}_{\mathrm{Ext}(\vec{\lambda},\vec{\mu})}[\mathrm{diag.}(\epsilon_1, \epsilon_2; \vec{a}, \vec{b})] = \sum_{i=1}^{|\vec{\lambda}|+|\vec{\mu}|} e^{s_i}, \tag{3.18}
$$

where each power s_i is a linear combination of $\epsilon_1, \epsilon_2, a_k$ and b_k. Then the basic ingredient in the following computation is

$$n_f^S[\vec{\lambda}, \vec{\mu}](\vec{a}, \vec{b}\ ; m) := \prod_{i=1}^{|\vec{\lambda}|+|\vec{\mu}|} (s_i - m), \tag{3.19}$$

which was originally denoted by z_{bif}^S in [7]. This is the contribution of the bifundamental matter hypermultiplet in the localization formula of the instanton partition function in the presence of the (full) surface operator.

To reformulate the instanton partition function Alday and Tachikawa [7] introduced a Hilbert space $\mathcal{H}_{\vec{a}}^S$ with basis $|\vec{\lambda}\rangle\rangle$. The inner product is defined by

$$\langle\langle\vec{\lambda}|\vec{\mu}\rangle\rangle = \frac{\delta_{\vec{\lambda},\vec{\mu}}}{n_{\mathrm{vec}}[\vec{\lambda}](\vec{a})}, \tag{3.20}$$

where $n_{\mathrm{vec}}[\vec{\lambda}](\vec{a}) := n_f^S[\vec{\lambda}, \vec{\lambda}](\vec{a}, \vec{a}\ ; 0)$. We will need the operator that counts the topological number:

$$\hat{K}_i|\vec{\lambda}\rangle\rangle = k_i(\vec{\lambda})|\vec{\lambda}\rangle\rangle. \tag{3.21}$$

Alday-Tachikawa also introduced the intertwining operator $\Phi^S_{\vec{a},m,\vec{b}} : \mathcal{H}^S_{\vec{b}} \longrightarrow \mathcal{H}^S_{\vec{a}}$, which is defined by

$$\Phi^S_{\vec{a},m,\vec{b}}|\vec{\lambda}\rangle\rangle_{\vec{b}} = \frac{1}{n_{\mathrm{vec}}[\vec{\lambda}](\vec{b})} \sum_{\vec{\mu}} n_f^S[\vec{\mu}, \vec{\lambda}](\vec{a}, \vec{b}\ ; m)|\vec{\mu}\rangle\rangle_{\vec{a}}. \tag{3.22}$$

Then the instanton partition function with four flavors in the presence of the surface operator is given by the following 'vacuum' expectation value (eq. (3.21) of [7]):

$$Z^{(S)}_{\mathrm{N_f}=4}(a, M_i, \epsilon_1, \epsilon_2; x, y) = {}_{a_1}\langle\langle\vec{\emptyset}|\Phi^S_{a_1,m_1,a} x^{\hat{K}_1} y^{\hat{K}_2} \Phi^S_{a,m_2,a_2}|\vec{\emptyset}\rangle\rangle_{a_2}, \tag{3.23}$$

where x and y are formal parameters of topological (instanton-monopole) expansion. The mass of the hypermultiplets M_i gives the parameters a_i and m_i in AGT like fashion:

$$\begin{aligned} a_1 &= M_1 - M_2, \quad m_1 = M_1 + M_2, \\ a_2 &= M_3 - M_4, \quad m_2 = M_3 + M_4. \end{aligned} \tag{3.24}$$

The above reformulation is convenient for identifying the partition function with the conformal block on the sphere with four punctures[4]. Inserting a complete system of $\mathcal{H}^S_a$

[4]If we have an adjoint matter the partition function is given by the trace over $\mathcal{H}^S_{\vec{a}}$, since it should be identified with the conformal block on the torus with a single puncture.

in the intermediate channel we obtain

$$\begin{aligned} Z^{(S)}_{\mathrm{N_f}=4}(a, M_i, \epsilon_1, \epsilon_2; x, y) &= \sum_{\vec{\lambda}} x^{k_1(\vec{\lambda})} y^{k_2(\vec{\lambda})} (n_{\mathrm{vec}}[\vec{\lambda}](\vec{a}))\ {}_{a_1}\langle\!\langle \vec{\emptyset} | \Phi^S_{a_1,m_1,a} | \vec{\lambda} \rangle\!\rangle_a \cdot\ {}_a\langle\!\langle \vec{\lambda} | \Phi^S_{a,m_2,a_2} | \vec{\emptyset} \rangle\!\rangle_{a_2} \\ &= \sum_{\vec{\lambda}} x^{k_1(\vec{\lambda})} y^{k_2(\vec{\lambda})} \frac{n^S_f[\vec{\emptyset}, \vec{\lambda}](a_1, a\ ; m_1) \cdot n^S_f[\vec{\lambda}, \vec{\emptyset}](a, a_2\ ; m_2)}{n_{\mathrm{vec}}[\vec{\lambda}](\vec{a})}. \end{aligned} \quad (3.25)$$

Hence to compute $Z^{(S)}_{\mathrm{N_f}=4}(a, M_i, \epsilon_1, \epsilon_2; x, y)$ we only need the equivariant character where one of the pairs of partitions is trivial. In this case among eight types of contributions given in Appendix A only two terms survive, which give

$$\mathrm{Tr}_{\mathrm{Ext}(\vec{\lambda},\vec{\emptyset})}[g] = -\sum_{k\geq 1} e^{a_k - b_1} e^{\epsilon_1 - \epsilon_2 \lfloor \frac{k}{2} \rfloor} \frac{e^{-\epsilon_1 \lambda_{k,k}} - 1}{e^{\epsilon_1} - 1} - \sum_{k\geq 1} e^{a_{k+1} - b_2} e^{\epsilon_1 - \epsilon_2 \lfloor \frac{k}{2} - \frac{1}{2} \rfloor} \frac{e^{-\epsilon_1 \lambda_{k+1,k}} - 1}{e^{\epsilon_1} - 1}, \quad (3.26)$$

and

$$\mathrm{Tr}_{\mathrm{Ext}(\vec{\emptyset},\vec{\lambda})}[g] = \sum_{k\geq 1} e^{a_1 - b_{k+1}} e^{\epsilon_1 + \epsilon_2 \lfloor \frac{k}{2} + \frac{1}{2} \rfloor} \frac{e^{\epsilon_1 \lambda_{k+1,k}} - 1}{e^{\epsilon_1} - 1} + \sum_{k\geq 1} e^{a_2 - b_k} e^{\epsilon_1 + \epsilon_2 \lfloor \frac{k}{2} \rfloor} \frac{e^{\epsilon_1 \lambda_{k,k}} - 1}{e^{\epsilon_1} - 1}. \quad (3.27)$$

From (3.26) and (3.27) we obtain the data for $n^S_f[\vec{\lambda}, \vec{\emptyset}](a, b\ ; m)$, which leads the partition function:

$$\begin{aligned} &Z^{(S)}_{\mathrm{N_f}=4}(a, M_i, \epsilon_1, \epsilon_2; x, y) \\ &= 1 + \frac{(-a + \epsilon_1 - 2M_1)(a - 2M_3)}{\epsilon_1(-2a + \epsilon_1)} x + \frac{(a + \epsilon_1 + \epsilon_2 - 2M_2)(-a - 2M_4)}{\epsilon_1(2a + \epsilon_1 + \epsilon_2)} y \\ &\quad + \frac{(-a + \epsilon_1 - 2M_1)(-a + 2\epsilon_1 - 2M_1)(a - 2M_3)(a - \epsilon_1 - 2M_3)}{2{\epsilon_1}^2(-2a + \epsilon_1)(-2a + 2\epsilon_1)} x^2 \\ &\quad + \frac{(a + \epsilon_1 + \epsilon_2 - 2M_2)(a + 2\epsilon_1 + \epsilon_2 - 2M_2)(-a - 2M_4)(-a - \epsilon_1 - 2M_4)}{2{\epsilon_1}^2(2a + \epsilon_1 + \epsilon_2)(2a + 2\epsilon_1 + \epsilon_2)} y^2 \\ &\quad + \frac{(-a + \epsilon_1 - 2M_1)(-a + \epsilon_1 + \epsilon_2 - 2M_2)(a - 2M_3)(a - 2M_4)}{\epsilon_1 \epsilon_2 (2a)(-2a + \epsilon_1)} xy \\ &\quad + \frac{(-a + \epsilon_1 - 2M_1)(a + \epsilon_1 + \epsilon_2 - 2M_2)(a - 2M_3)(-a - 2M_4)}{{\epsilon_1}^2(-2a)(2a + \epsilon_2)} xy \\ &\quad + \frac{(a + \epsilon_1 + \epsilon_2 - 2M_1)(a + \epsilon_1 + \epsilon_2 - 2M_2)(-a - \epsilon_2 - 2M_3)(-a - 2M_4)}{\epsilon_1 \epsilon_2 (2a + \epsilon_1 + \epsilon_2)(-2a - \epsilon_2)} xy \\ &\quad + \cdots\cdots . \end{aligned} \quad (3.28)$$

The instanton partition functions with surface operator for asymptotically free theories with $N_f \leq 3$ can be obtained by the decoupling limit, where the expansion parameters

x, y are promoted to Λ_1, Λ_2 with appropriate mass dimension. There are several choices of the set of M_i's with $M_i \to \infty$. In any case one of the characteristic features of the decoupling is that not only the denominators but also the numerators of the x^2 and y^2 terms are of factorized form. This is because there is only one fixed point with the corresponding topological number. In [7] it is observed that up to an appropriate $U(1)$ factor the above partition function (3.28) coincides with the four-point conformal block of $SL(2)$ current algebra on the sphere with an insertion of the operator $\mathcal{K}$ which was introduced in [7]. In sections 4 and 5 we will explicitly check that the partition function (3.28) and its decoupling limit also agree with the Liouville correlation functions on the sphere with a degenerate field insertion; see subsection 5.5 for a summary and a rule of the correspondence.

One can check that the free energy $F^{(S)}_{\mathrm{N_f}=4}(a, M_i, \epsilon_1, \epsilon_2) = \log Z^{(S)}_{\mathrm{N_f}=4}(a, M_i, \epsilon_1, \epsilon_2)$ has a correct pole structure, namely the poles of x^2 and y^2 terms are ϵ_1^{-1}, while that of xy term is $(\epsilon_1\epsilon_2)^{-1}$. However, the explicit form is rather lengthy. We only quote the lowest terms:

$$\begin{aligned}
&\lim_{\epsilon_1,\epsilon_2\to 0} \epsilon_1 \cdot F^{(S)}_{\mathrm{N_f}=4}(a, M_i, \epsilon_1, \epsilon_2)|_{x^2} = -\frac{1}{16}\frac{(a+2M_1)(a-2M_3)(3a^2+2a(M_1-M_3)+4M_1M_3)}{a^3} \\
&\lim_{\epsilon_1,\epsilon_2\to 0} \epsilon_1 \cdot F^{(S)}_{\mathrm{N_f}=4}(a, M_i, \epsilon_1, \epsilon_2)|_{y^2} = \frac{1}{16}\frac{(a-2M_2)(a+2M_4)(3a^2+2a(M_4-M_2)+4M_2M_4)}{a^3} \\
&\lim_{\epsilon_1,\epsilon_2\to 0} \epsilon_1\epsilon_2 \cdot F^{(S)}_{\mathrm{N_f}=4}(a, M_i, \epsilon_1, \epsilon_2)|_{xy} \\
&\quad = \frac{-a^4 + 4(M_1M_3 + M_1M_4 + M_2M_3 + M_2M_4 - M_1M_2 - M_3M_4)a^2 - 16M_1M_2M_3M_4}{2a^2}.
\end{aligned} \tag{3.29}$$

4 CFT correlation functions with degenerate field insertion

In [3] it was claimed that the surface operator $\mathcal{S} \subset \mathbb{R}^4$ in the supersymmetric gauge theory with eight supercharges corresponds to the degenerate primary operator $\Phi_{1,2}(z)$ in the Liouville CFT. An explanation of the correspondence from the viewpoint of the M2/M5-brane system was also given. Since the operator $\Phi_{1,2}(z)$ which has the momentum $-\frac{1}{2b}$ satisfies the null state condition $(b^2L_{-1}^2 + L_{-2})\Phi_{1,2}(z) = 0$, when it is inserted in any

CFT correlation functions, we have

$$b^2\partial_z^2\Phi_{1,2}(z) = - : T(z)\Phi_{1,2}(z) :, \tag{4.1}$$

where $T(z)$ is the energy momentum tensor and $:\quad:$ denotes the normal ordering. When the operator $\Phi_{1,2}(z)$ is inserted, the correlation function has an additional dependence on the position z of the degenerate operator. One of the points in [3] is that this dependence appears in the subleading term of the semi-classical approximation:

$$\begin{aligned}\Psi(a_i, z) &:= \langle \Phi_{1,2}(z) V_{m_1}(z_1) \cdots V_{m_n}(z_n)\rangle_{\{a_i\}} \\ &\sim \exp\left(-\frac{\mathcal{F}(a_i)}{\hbar^2} + \frac{\mathcal{W}(a_i, z)}{b\hbar} + \cdots \right).\end{aligned} \tag{4.2}$$

Recall that the original observations of [2] are that

$$Z(a_i) := \langle V_{m_1}(z_1) \cdots V_{m_n}(z_n)\rangle_{\{a_i\}} \sim \exp\left(-\frac{\mathcal{F}(a_i)}{\hbar^2} + \cdots \right) \tag{4.3}$$

coincides with the Nekrasov partition function and that

$$\langle T(z) V_{m_1}(z_1) \cdots V_{m_n}(z_n)\rangle_{\{a_i\}} \sim -\frac{1}{\hbar^2}\phi^{\mathrm{SW}}(z) \langle V_{m_1}(z_1) \cdots V_{m_n}(z_n)\rangle_{\{a_i\}}, \tag{4.4}$$

where $x^2 = \phi^{\mathrm{SW}}(z)$ gives the Seiberg-Witten curve which is a double covering of the punctured Riemann sphere. In asymptotically free theories we should consider the correlation functions with respect to the state introduced by Gaiotto [13]. It is natural to call them irregular conformal blocks, since the differential equations for such correlation functions have irregular singularities in general. In the appendix of [17] it was noticed that the differential equation for irregular conformal blocks with $\Phi_{1,2}(z)$ insertion coincides with the differential equation for the instanton partition function with parabolic structure derived in [16]. Based on these works we expect that the instanton partition functions computed in section 3 by localization formula are obtained from the one point function of $\Phi_{1,2}(z)$ with respect to the Gaiotto state. In this section we check the correspondence for $N_f = 0, 1$ and 2 (see also the next section for the discussion by degenerations from the superconformal theory with $N_f = 4$). In section 4.4 we consider the multi-point irregular conformal blocks which should correspond to the instanton partition functions with multi-surface operators.

4.1 Pure $SU(2)$

Let us consider the (normalized) correlation function

$$Z^{(0)}_{\mathrm{null}}(z,a,\Lambda) := \frac{\langle \Delta_-, \Lambda | \Phi_{1,2}(z) | \Delta_+, \Lambda \rangle}{\langle \Delta_a, \Lambda | \Delta_a, \Lambda \rangle} =: \frac{\Psi^{(0)}(z,a,\Lambda)}{Z^{(0)}_c(a,\Lambda)}, \quad \Delta_\pm := \Delta(a \pm \frac{1}{4b}), \tag{4.5}$$

where $\Delta_a := \Delta(a) := (b+b^{-1})^2/4 - a^2$ is the conformal dimension and $\Phi_{1,2}(z)$ is the degenerate primary field with the conformal dimension $h_{1,2} = -1/2 - 3/(4b^2)$. The Gaiotto state $|\Delta, \Lambda\rangle$ is the state in the Virasoro Verma module $V(\Delta, c)$ with the conformal dimension Δ and the central charge $c = 1 + 6(b+b^{-1})^2$, which is characterized by

$$L_1 |\Delta, \Lambda\rangle = \Lambda^2 |\Delta, \Lambda\rangle, \quad L_2 |\Delta, \Lambda\rangle = 0. \tag{4.6}$$

There is an ambiguity in the choice of the conformal weights $\Delta_\pm := \Delta(\alpha_\pm)$ of the Gaiotto state. According to the fusion rule of $\Phi_{1,2}$ operator, $\langle \Delta_-, \Lambda | \Phi_{1,2}(z) | \Delta_+, \Lambda \rangle$ is non-vanishing if and only if $\alpha_+ - \alpha_- = \pm \frac{1}{2b}$. The above choice $\alpha_\pm = a \pm \frac{1}{4b}$ is the symmetric one, which leads a result that is invariant under $a \to -a$.

In [13] it was conjectured that $Z^{(0)}_c(a,\Lambda)$ coincides with the Nekrasov partition function in the pure $SU(2)$ supersymmetric gauge theory, which has been proved in [48]. According to the appendix of [17], by putting $\Psi^{(0)}(z,a,\Lambda) = z^{\Delta_- - \Delta_+ - h_{1,2}} Y^{(0)}(z,a,\Lambda)$, one obtains the second order differential equation[5]:

$$\left[\left(bz \frac{\partial}{\partial z} \right)^2 + 2abz \frac{\partial}{\partial z} + \Lambda^2 (z + z^{-1}) + \frac{\Lambda}{4} \frac{\partial}{\partial \Lambda} \right] Y^{(0)}(z,a,\Lambda) = 0. \tag{4.7}$$

Since we want to compare the instanton partition function computed in the previous section by localization theorem with solutions to the differential equation (4.7), we look for a solution of the form

$$Y^{(0)}(z,a,\Lambda) = \sum_{n=0}^{\infty} \Lambda^{2n} Y^{(0)}_n(z,a) \tag{4.8}$$

with the initial condition $Y^{(0)}_0(z,a) = 1$. It is convenient to introduce a mass scale $\hbar$ and scale the parameters as follows:

$$a \longrightarrow \frac{a}{\hbar}, \quad \Lambda \longrightarrow \frac{\Lambda}{\hbar}. \tag{4.9}$$

[5]Note that the operator $\Phi_{1,2}(z)$ in this paper corresponds to $\Phi_{2,1}(z)$ in the convention of [17].

We also introduce the parameters $\epsilon_1 = b\hbar$, $\epsilon_2 = b^{-1}\hbar$ corresponding to the parameters of the Ω background of Nekrasov. Then we find the following differential equations for the coefficients $Y_n^{(0)}(z,a)$ in the expansion (4.8),

$$\left[\left(\epsilon_1 z\frac{\partial}{\partial z}\right)^2 + 2a\epsilon_1 z\frac{\partial}{\partial z} + \frac{n}{2}\epsilon_1\epsilon_2\right] Y_n^{(0)}(z,a) + (z+z^{-1})Y_{n-1}^{(0)}(z,a) = 0, \qquad n \geq 1. \quad (4.10)$$

A power series solution to (4.10) is given by

$$Y_n^{(0)}(z,a) = \sum_{k=-\infty}^{\infty} A_{n,k}^{(0)} z^k, \qquad A_{0,k}^{(0)} = \delta_{0,k}, \quad A_{n,k}^{(0)} = -\frac{A_{n-1,k-1}^{(0)} + A_{n-1,k+1}^{(0)}}{\epsilon_1\left(2ak + \epsilon_1 k^2 + \frac{1}{2}n\epsilon_2\right)}, \quad (4.11)$$

and we find the following lower order terms in the expansion (4.8),

$$Y_1^{(0)}(z,a) = -\frac{1}{\epsilon_1(-2a+\epsilon_1+\frac{\epsilon_2}{2})z} - \frac{z}{\epsilon_1(2a+\epsilon_1+\frac{\epsilon_2}{2})}, \quad (4.12)$$

$$\begin{aligned} Y_2^{(0)}(z,a) = &\frac{1}{2\epsilon_1^2(-2a+\epsilon_1+\frac{\epsilon_2}{2})(-2a+2\epsilon_1+\frac{\epsilon_2}{2})z^2} + \frac{2\epsilon_1+\epsilon_2}{\epsilon_1^2\epsilon_2(-2a+\epsilon_1+\frac{\epsilon_2}{2})(2a+\epsilon_1+\frac{\epsilon_2}{2})} \\ &+\frac{z^2}{2\epsilon_1^2(2a+\epsilon_1+\frac{\epsilon_2}{2})(2a+2\epsilon_1+\frac{\epsilon_2}{2})}, \end{aligned} \quad (4.13)$$

$$\begin{aligned} Y_3^{(0)}(z,a) = &-\frac{1}{6\epsilon_1^3(-2a+\epsilon_1+\frac{\epsilon_2}{2})(-2a+2\epsilon_1+\frac{\epsilon_2}{2})(-2a+3\epsilon_1+\frac{\epsilon_2}{2})z^3} \\ &-\frac{16\epsilon_1^2+14\epsilon_1\epsilon_2+3\epsilon_2^2-16a\epsilon_1-4a\epsilon_2}{4\epsilon_1^3\epsilon_2(-2a+\epsilon_1+\frac{\epsilon_2}{2})(-2a+2\epsilon_1+\frac{\epsilon_2}{2})(-2a+\epsilon_1+\frac{3\epsilon_2}{2})(2a+\epsilon_1+\frac{\epsilon_2}{2})z} + \cdots . \end{aligned} \quad (4.14)$$

If we make the shift that was discussed in the last section to make the partition function invariant under the $SU(2)$ Weyl transformation $a \to -a$, then (4.12) – (4.14) completely agree to the instanton expansion (3.10) of the partition function of pure $SU(2)$ theory. The free energy is defined by

$$F_{\text{null}}^{(0)}(z,a,\Lambda) := \log Z_{\text{null}}^{(0)}(z,a,\Lambda) = \left(\frac{a}{\epsilon_1} + \frac{1}{2} + \frac{3\epsilon_2}{4\epsilon_1}\right)\log z + \log\frac{1+\sum_{n=1}^{\infty}\Lambda^{2n}Y_n^{(0)}(z,a)}{Z_c^{(0)}(a,\Lambda)}, \quad (4.15)$$

and then we obtain

$$F_{\text{null}}^{(0)}(z,a,\Lambda) = \frac{1}{\epsilon_1}\left\{F_{-1}^{(0)} + \left(\epsilon_1 F_{0,1}^{(0)} + \epsilon_2 F_{0,2}^{(0)}\right) + \mathcal{O}(\epsilon^2)\right\}, \quad (4.16)$$

where the leading term is

$$F_{-1}^{(0)} = \log z^a - \frac{z^2-1}{2az}\Lambda^2 - \frac{z^4-1}{16a^3z^2}\Lambda^4 - \frac{z^6+3z^4-3z^2-1}{48a^5z^3}\Lambda^6 - \frac{5z^8+16z^6-16z^2-5}{512a^7z^4}\Lambda^8 + \cdots . \quad (4.17)$$

4.2 $SU(2)$ with one fundamental matter

Next, we consider the correlation function

$$Z_{\rm null}^{(1)}(z,a,m,\Lambda) := \frac{\langle \Delta_-,\Lambda,m|\Phi_{1,2}(z)|\Delta_+,\Lambda\rangle}{\langle \Delta_a,\Lambda,m|\Delta_a,\Lambda\rangle} =: \frac{\Psi^{(1)}(z,a,m,\Lambda)}{Z_c^{(1)}(a,m,\Lambda)}, \tag{4.18}$$

where the Gaiotto state $|\Delta,\Lambda,m\rangle$ in the Virasoro Verma module $V(\Delta,c)$ satisfies

$$L_2|\Delta,\Lambda,m\rangle = -\Lambda^2|\Delta,\Lambda,m\rangle, \quad L_1|\Delta,\Lambda\rangle = -2m\Lambda|\Delta,\Lambda,m\rangle. \tag{4.19}$$

The denominator $Z_c^{(1)}(a,m,\Lambda)$ of (4.18) coincides with the Nekrasov partition function of $SU(2)$ supersymmetric gauge theory with one fundamental matter [13, 48]. By putting $\Psi^{(1)}(z,a,m,\Lambda) = z^{\Delta_- - \Delta_+ - h_{1,2}} Y^{(1)}(z,a,m,\Lambda)$, one obtains the second order differential equation,

$$\left[\left(bz\frac{\partial}{\partial z}\right)^2 + \left(2ab+\frac{1}{6}\right)z\frac{\partial}{\partial z} - \Lambda^2(z^2-z^{-1}) - 2m\Lambda z + \frac{\Lambda}{3}\frac{\partial}{\partial\Lambda}\right] Y^{(1)}(z,a,m,\Lambda) = 0. \tag{4.20}$$

By the decoupling limit of the matter $m\to\infty, -2m\Lambda^3\to\Lambda^4, 4m^2z^3\to\Lambda^2z^3$, the differential equation (4.20) is reduced to (4.7). After introducing the mass scale $\hbar$ as (4.9) and $m\to m/\hbar$, we obtain the following solution to (4.20),

$$Y^{(1)}(z,a,m,\Lambda) = \sum_{n=0}^{\infty}\Lambda^n Y_n^{(1)}(z,a,m), \quad Y_0^{(1)}(z,a,m) = 1, \tag{4.21}$$

$$Y_n^{(1)}(z,a,m) = \sum_{k=-\infty}^{\infty} A_{n,k}^{(1)} z^k,$$

$$A_{0,k}^{(1)} = \delta_{0,k}, \quad A_{n,k}^{(1)} = \frac{A_{n-2,k-2}^{(1)} - A_{n-2,k+1}^{(1)} + 2mA_{n-1,k-1}^{(1)}}{\epsilon_1\left((2a+\frac{1}{6}\epsilon_2)k + \epsilon_1 k^2 + \frac{1}{3}n\epsilon_2\right)}. \tag{4.22}$$

Lower order terms in the expansion (4.21) are given by

$$Y_1^{(1)}(z,a,m) = \frac{2mz}{\epsilon_1(2a+\epsilon_1+\frac{\epsilon_2}{2})}, \tag{4.23}$$

$$Y_2^{(1)}(z,a,m) = \frac{-1}{\epsilon_1(-2a+\epsilon_1+\frac{\epsilon_2}{2})z} + \frac{(2\epsilon_1^2+\epsilon_1\epsilon_2+4a\epsilon_1+8m^2)z^2}{4\epsilon_1^2(2a+\epsilon_1+\frac{\epsilon_2}{2})(2a+2\epsilon_1+\frac{\epsilon_2}{2})}, \tag{4.24}$$

$$\begin{aligned} Y_3^{(1)}(z,a,m) = &\frac{-2m(2\epsilon_1+\epsilon_2)}{\epsilon_1^2\epsilon_2(-2a+\epsilon_1+\frac{\epsilon_2}{2})(2a+\epsilon_1+\frac{\epsilon_2}{2})} \\ &+\frac{m(10\epsilon_1^2+3\epsilon_1\epsilon_2+12a\epsilon_1+8m^2)z^3}{6\epsilon_1^3(2a+\epsilon_1+\frac{\epsilon_2}{2})(2a+2\epsilon_1+\frac{\epsilon_2}{2})(2a+3\epsilon_1+\frac{\epsilon_2}{2})}. \end{aligned} \tag{4.25}$$

Mimicking the prescription for the Nekrasov partition function [2], we multiply $Y^{(1)}(z,a,m,\Lambda)$ by an overall factor $\exp(-\Lambda z/\epsilon_1)$,

$$\widetilde{Y}^{(1)}(z,a,m,\Lambda) := e^{-\frac{\Lambda}{\epsilon_1}z} Y^{(1)}(z,a,m,\Lambda) = \sum_{n=0}^{\infty} \Lambda^n \widetilde{Y}^{(1)}_n(z,a,m), \tag{4.26}$$

to obtain[6]

$$\widetilde{Y}^{(1)}_1(z,a,m) = \frac{(2m-2a-\epsilon_1-\frac{\epsilon_2}{2})z}{\epsilon_1(2a+\epsilon_1+\frac{\epsilon_2}{2})}, \tag{4.27}$$

$$\widetilde{Y}^{(1)}_2(z,a,m) = \frac{-1}{\epsilon_1(-2a+\epsilon_1+\frac{\epsilon_2}{2})z} + \frac{(2m-2a-\epsilon_1-\frac{\epsilon_2}{2})(2m-2a-3\epsilon_1-\frac{\epsilon_2}{2})z^2}{2\epsilon_1^2(2a+\epsilon_1+\frac{\epsilon_2}{2})(2a+2\epsilon_1+\frac{\epsilon_2}{2})}, \tag{4.28}$$

$$\begin{aligned}\widetilde{Y}^{(1)}_3(z,a,m) = &-\frac{2m(2\epsilon_1+\epsilon_2)-2a\epsilon_2-\epsilon_1\epsilon_2-\frac{\epsilon_2^2}{2}}{\epsilon_1^2\epsilon_2(-2a+\epsilon_1+\frac{\epsilon_2}{2})(2a+\epsilon_1+\frac{\epsilon_2}{2})} \\ &+\frac{(2m-2a-\epsilon_1-\frac{\epsilon_2}{2})(2m-2a-3\epsilon_1-\frac{\epsilon_2}{2})(2m-2a-5\epsilon_1-\frac{\epsilon_2}{2})z^3}{6\epsilon_1^3(2a+\epsilon_1+\frac{\epsilon_2}{2})(2a+2\epsilon_1+\frac{\epsilon_2}{2})(2a+3\epsilon_1+\frac{\epsilon_2}{2})}. \end{aligned}\tag{4.29}$$

We find an agreement with the computation in the gauge theory. Namely (4.27) – (4.29) is consistent with the decoupling limit of the partition function (3.28). Especially the numerators of the coefficients of z^2 and z^3 are of factorized form, which is not the case in (4.23) – (4.25). The multiplication of the overall factor is crucial for this factorization, which is a feature of the localization computation formula on the gauge theory side. As in the case of pure Yang-Mills theory, the free energy is

$$F^{(1)}_{\text{null}}(z,a,m,\Lambda) := \log Z^{(1)}_{\text{null}}(z,a,m,\Lambda) = \frac{1}{\epsilon_1}\left\{F^{(1)}_{-1} + \left(\epsilon_1 F^{(1)}_{0,1} + \epsilon_2 F^{(1)}_{0,2}\right) + \mathcal{O}(\epsilon^2)\right\} \tag{4.30}$$

and the leading term is

$$\begin{aligned}F^{(1)}_{-1} = &\log z^a + \frac{mz}{a}\Lambda + \frac{(a^2-m^2)z^3+2a^2}{4a^3 z}\Lambda^2 - \frac{m(a^2-m^2)z^3}{6a^5}\Lambda^3 \\ &+\left\{-\frac{(a^2-m^2)(a^2-5m^2)z^4}{32a^7} + \frac{(a^2-m^2)z}{4a^5} + \frac{1}{16a^3z^2}\right\}\Lambda^4 + \cdots . \end{aligned}\tag{4.31}$$

4.3 $SU(2)$ with two fundamental matters (first realization)

Let us concentrate on the first realization [47, 13] of $SU(2)$ theory with two fundamental matters and consider the correlation function

$$Z^{(2)}_{\text{null}}(z,a,m_i,\Lambda) := \frac{\langle \Delta_-, \Lambda, m_2 | \Phi_{1,2}(z) | \Delta_+, \Lambda, m_1\rangle}{\langle \Delta_a, \Lambda, m_2 | \Delta_a, \Lambda, m_1\rangle} =: \frac{\Psi^{(2)}(z,a,m_i,\Lambda)}{Z^{(2)}_c(a,m_i,\Lambda)}, \tag{4.32}$$

[6]The origin of the overall factor is made clear in the next section where we discuss the decoupling limit at the level of differential equations.

where by multiplying an overall factor, $\exp(-2\Lambda^2/\epsilon_1\epsilon_2)\cdot Z_c^{(2)}(a,m_i,\Lambda)$ coincides with the Nekrasov partition function after the scaling $\Lambda\to 2\Lambda$ [13, 48]. In parallel with the above computations, by putting $\Psi^{(2)}(z,a,m_i,\Lambda)=z^{\Delta_- -\Delta_+ -h_{1,2}}Y^{(2)}(z,a,m_i,\Lambda)$, one obtains the second order differential equation,

$$\left[\left(bz\frac{\partial}{\partial z}\right)^2+2abz\frac{\partial}{\partial z}-\Lambda^2(z^2+z^{-2})-2\Lambda(m_2z+m_1z^{-1})+\frac{\Lambda}{2}\frac{\partial}{\partial\Lambda}\right]Y^{(2)}(z,a,m_i,\Lambda)=0, \tag{4.33}$$

where by the decoupling limit of the two fundamental matters $m_i\to\infty, m_i\Lambda\to-\Lambda^2/2$, we see that the differential equation (4.33) is reduced to (4.7). By introducing the mass scale $\hbar$ as (4.9) and $m_i\to m_i/\hbar$, we obtain the following solution to (4.33),

$$Y^{(2)}(z,a,m_i,\Lambda)=\sum_{n=0}^{\infty}\Lambda^nY_n^{(2)}(z,a,m_i),\quad Y_0^{(2)}(z,a,m_i)=1, \tag{4.34}$$

$$Y_n^{(2)}(z,a,m_i)=\sum_{k=-\infty}^{\infty}A_{n,k}^{(2)}z^k,$$

$$A_{0,k}^{(2)}=\delta_{0,k},\quad A_{n,k}^{(2)}=\frac{A_{n-2,k-2}^{(2)}+A_{n-2,k+2}^{(2)}+2(m_2A_{n-1,k-1}^{(2)}+m_1A_{n-1,k+1}^{(2)})}{\epsilon_1\left(2ak+\epsilon_1k^2+\frac{1}{2}n\epsilon_2\right)}, \tag{4.35}$$

with

$$Y_1^{(2)}(z,a,m_i)=\frac{2m_1}{\epsilon_1(-2a+\epsilon_1+\frac{\epsilon_2}{2})z}+\frac{2m_2z}{\epsilon_1(2a+\epsilon_1+\frac{\epsilon_2}{2})}, \tag{4.36}$$

$$\begin{aligned}Y_2^{(2)}(z,a,m_i)&=\frac{2\epsilon_1^2+\epsilon_1\epsilon_2-4a\epsilon_1+8m_1^2}{4\epsilon_1^2(-2a+\epsilon_1+\frac{\epsilon_2}{2})(-2a+2\epsilon_1+\frac{\epsilon_2}{2})z^2}\\&\quad+\frac{4m_1m_2(2\epsilon_1+\epsilon_2)}{\epsilon_1^2\epsilon_2(-2a+\epsilon_1+\frac{\epsilon_2}{2})(2a+\epsilon_1+\frac{\epsilon_2}{2})}+\frac{(2\epsilon_1^2+\epsilon_1\epsilon_2+4a\epsilon_1+8m_2^2)z^2}{4\epsilon_1^2(2a+\epsilon_1+\frac{\epsilon_2}{2})(2a+2\epsilon_1+\frac{\epsilon_2}{2})},\end{aligned} \tag{4.37}$$

$$Y_3^{(2)}(z,a,m_i)=\frac{m_1(10\epsilon_1^2+3\epsilon_1\epsilon_2-12a\epsilon_1+8m_1^2)}{6\epsilon_1^3(-2a+\epsilon_1+\frac{\epsilon_2}{2})(-2a+2\epsilon_1+\frac{\epsilon_2}{2})(-2a+3\epsilon_1+\frac{\epsilon_2}{2})z^3}+\cdots. \tag{4.38}$$

As before, multiplying $Y^{(2)}(z,a,m_i,\Lambda)$ by an overall factor $\exp(-\Lambda(z+z^{-1})/\epsilon_1-2\Lambda^2/\epsilon_1\epsilon_2)$,

$$\widetilde{Y}^{(2)}(z,a,m_i,\Lambda):=e^{-\frac{\Lambda}{\epsilon_1}(z+z^{-1})}e^{-\frac{2\Lambda^2}{\epsilon_1\epsilon_2}}Y^{(2)}(z,a,m_i,\Lambda)=\sum_{n=0}^{\infty}\Lambda^n\widetilde{Y}_n^{(2)}(z,a,m_i), \tag{4.39}$$

we arrive at

$$\widetilde{Y}_1^{(2)}(z,a,m_i) = \frac{2m_1+2a-\epsilon_1-\frac{\epsilon_2}{2}}{\epsilon_1(-2a+\epsilon_1+\frac{\epsilon_2}{2})z} + \frac{(2m_2-2a-\epsilon_1-\frac{\epsilon_2}{2})z}{\epsilon_1(2a+\epsilon_1+\frac{\epsilon_2}{2})}, \tag{4.40}$$

$$\widetilde{Y}_2^{(2)}(z,a,m_i) = \frac{(2m_1+2a-\epsilon_1-\frac{\epsilon_2}{2})(2m_1+2a-3\epsilon_1-\frac{\epsilon_2}{2})}{2\epsilon_1^2(-2a+\epsilon_1+\frac{\epsilon_2}{2})(-2a+2\epsilon_1+\frac{\epsilon_2}{2})z^2}$$
$$+\frac{(2m_2-2a-\epsilon_1-\frac{\epsilon_2}{2})(2m_2-2a-3\epsilon_1-\frac{\epsilon_2}{2})z^2}{2\epsilon_1^2(2a+\epsilon_1+\frac{\epsilon_2}{2})(2a+2\epsilon_1+\frac{\epsilon_2}{2})} + \cdots, \tag{4.41}$$

$$\widetilde{Y}_3^{(2)}(z,a,m_i) = \frac{(2m_1+2a-\epsilon_1-\frac{\epsilon_2}{2})(2m_1+2a-3\epsilon_1-\frac{\epsilon_2}{2})(2m_1+2a-5\epsilon_1-\frac{\epsilon_2}{2})}{6\epsilon_1^3(-2a+\epsilon_1+\frac{\epsilon_2}{2})(-2a+2\epsilon_1+\frac{\epsilon_2}{2})(-2a+3\epsilon_1+\frac{\epsilon_2}{2})z^3} + \cdots. \tag{4.42}$$

Again we find an agreement with the decoupling limit of the partition function (3.28) computed by the localization theorem on the gauge theory side. The multiplication of the overall factor makes the numerator factorized. The free energy is

$$F_{\text{null}}^{(2)}(z,a,m_i,\Lambda) := \log Z_{\text{null}}^{(2)}(z,a,m_i,\Lambda) = \frac{1}{\epsilon_1}\left\{F_{-1}^{(2)} + \left(\epsilon_1 F_{0,1}^{(2)} + \epsilon_2 F_{0,2}^{(2)}\right) + \mathcal{O}(\epsilon^2)\right\}, \tag{4.43}$$

where the leading term is

$$F_{-1}^{(2)} = \log z^a - \frac{m_1 - m_2 z^2}{az}\Lambda + \frac{m_1^2 - a^2 + (a^2 - m_2^2)z^4}{4a^3 z^2}\Lambda^2 + \cdots. \tag{4.44}$$

4.4 Multi-point irregular conformal block

In the above computations, we explicitly checked that a single surface operator in $SU(2)$ gauge theories corresponds to the degenerate primary operator $\Phi_{1,2}(z)$ on the CFT side. It is natural to expect that the multi-surface operators correspond to the multi-degenerate primary operators $\Phi_{1,2}(z_1),\ldots,\Phi_{1,2}(z_h)$. Here we introduce the multi-point irregular conformal blocks which are to be compared with the computations in the B model in sections 5 and 6 (see also Appendix B for more detail). Let us consider

$$Z_{\text{null}}(z_1,\ldots,z_h) := \frac{\langle G'|\Phi_{1,2}(z_1)\cdots\Phi_{1,2}(z_h)|G''\rangle}{\langle G|G\rangle} =: \frac{\Psi(z_1,\ldots,z_h)}{Z_c}, \tag{4.45}$$

where $|G\rangle$ is the Gaiotto state that reproduces the Nekrasov partition function Z_c. The states $|G'\rangle$ and $|G''\rangle$ in the numerator should have shifted a parameters in order to be consistent with the fusion rule of the $\Phi_{1,2}$ operator. In Appendix B.1 $\Psi(z_1,z_2)$ and

$\Psi(z_1, z_2, z_3)$ for $N_f = 0$ theory are computed by solving the differential equation and in Appendix B.2 $\Psi(z_1, z_2)$ is computed for $N_f = 1$ theory.

After the scaling (4.9) we consider the self-dual case $\epsilon_1 = -\epsilon_2 = i\hbar$ and define the free energy

$$F_{\mathrm{null}}(z_1, \ldots, z_h) = \log Z_{\mathrm{null}}(z_1, \ldots, z_h) = \sum_{k=-1}^{\infty} \hbar^k F_{\mathrm{CFT}}^{(k)}(z_1, \ldots, z_h), \tag{4.46}$$

since the B model computations in sections 5 and 6 only provide the free energy with the self-dual Ω background. Some of the explicit computation are provided in Appendix B.

5 Degeneration scheme of CFT differential equations

5.1 Ward identities

In general, the N (or $N+2$)-points block on the Riemann sphere

$$\Psi(z_1, \ldots, z_N) = \langle A|\mathcal{O}_{h_1}(z_1) \cdots \mathcal{O}_N(z_N)|B\rangle, \tag{5.1}$$

can be determined by the Ward identity

$$\sum_{\text{all poles}} \langle A|\mathrm{Res}\Big(\xi(x)T(x)dx\Big)\mathcal{O}_1(z_1) \cdots \mathcal{O}_N(z_N)|B\rangle = 0, \tag{5.2}$$

where $\xi = \xi(x)\frac{\partial}{\partial x}$ is any rational vector field. For example, choosing the vector field ξ as $\xi(x) = \frac{z_1 x}{z_1 - x}$, one has

$$\begin{aligned}
&\Big[\Big(L_0 + \frac{1}{z_1}L_1 + \frac{1}{z_1^2}L_2 + \cdots\Big)_0 + \Big(-z_1^2 L_{-2} - z_1 L_{-1}\Big)_{z_1} \\
&+ \sum_{j=2}^{N}\Big(\frac{z_1 z_j}{z_1 - z_j}L_{-1} + \frac{z_1^2}{(z_1 - z_j)^2}L_0 + \frac{z_1^2}{(z_1 - z_j)^3}L_1 + \cdots\Big)_{z_j} \\
&+ \Big(z_1 L_{-1} + z_1^2 L_{-2} + z_1^3 L_{-3} + \cdots\Big)_{\infty}\Big]\Psi = 0,
\end{aligned} \tag{5.3}$$

where $(\cdots)_z$ means the action of $T(x)$ at z defined by

$$(L_n)_z\langle \cdots \mathcal{O}(z) \cdots\rangle = \oint_{x=z} \frac{dx}{2\pi i}(x-z)^{n+1}\langle \cdots T(x)\mathcal{O}(z) \cdots\rangle. \tag{5.4}$$

In the case where the operator $\mathcal{O}_1$ is the degenerate field $\Phi_{1,2}$ such that

$$L_{-2}\Phi_{1,2} = -b^2 L_{-1}^2 \Phi_{1,2}, \tag{5.5}$$

and other $\mathcal{O}_j$ are primaries with dimension h_j, then we have

$$\begin{aligned}&\Big[\Big(L_0+\frac{1}{z_1}L_1+\frac{1}{z_1^2}L_2+\cdots\Big)_0+\Big(b^2z_1^2\partial_{z_1}^2-z_1\partial_{z_1}\Big)\\&+\sum_{j=2}^{N}\Big(\frac{z_1z_j}{z_1-z_j}\partial_{z_j}+\frac{z_1^2}{(z_1-z_j)^2}h_j\Big)+\Big(z_1L_{-1}+z_1^2L_{-2}+z_1^3L_{-3}+\cdots\Big)_\infty\Big]\Psi=0.\end{aligned}\tag{5.6}$$

5.2 Differential equations

We will give a list of CFT differential equations with single $\Phi_{1,2}(z)$ operator insertion (Fig.2).

• $N_f=0$: $\Psi(z,\Lambda)=\big\langle\Delta_-,\Lambda\big|\Phi_{1,2}(z)\big|\Delta_+,\Lambda\big\rangle$,

$$b^2z^2\Psi_{zz}-\frac{3}{2}z\Psi_z+\frac{\Lambda}{4}\Psi_\Lambda+\Big(\frac{\Delta_-+\Delta_+-h_{1,2}}{2}+\Lambda^2(z+\frac{1}{z})\Big)\Psi=0.\tag{5.7}$$

• $N_f=1$: $\Psi(z,\Lambda)=\big\langle\Delta_-,\Lambda,m\big|\Phi_{1,2}(z)\big|\Delta_+,\Lambda\big\rangle$,

$$b^2z^2\Psi_{zz}-\frac{4}{3}z\Psi_z+\frac{\Lambda}{3}\Psi_\Lambda+\Big(\frac{\Delta_-+2\Delta_+-h_{1,2}}{3}+\Lambda^2(\frac{1}{z}-z^2)-2\Lambda mz\Big)\Psi=0.\tag{5.8}$$

• $N_f=2^{(1)}$ (1st realization): $\Psi(z,\Lambda)=\big\langle\Delta_-,\Lambda,m_2\big|\Phi_{1,2}(z)\big|\Delta_+,\Lambda,m_1\big\rangle$,

$$b^2z^2\Psi_{zz}-\frac{3}{2}z\Psi_z+\frac{\Lambda}{2}\Psi_\Lambda+\Big(\frac{\Delta_-+\Delta_+-h_{1,2}}{2}-\Lambda^2(z^2+\frac{1}{z^2})-2\Lambda(\frac{m_1}{z}+m_2z)\Big)\Psi=0.\tag{5.9}$$

• $N_f=2^{(2)}$ (2nd realization): $\Psi(z,\Lambda)=\big\langle\Delta_{m_+}\big|\Phi_{\Delta_{m_-}}(1)\Phi_{1,2}(z)\big|\Delta_+,\Lambda\big\rangle$,

$$b^2z(z-1)\Psi_{zz}-(2z-1)\Psi_z-\frac{\Lambda}{2z}\Psi_\Lambda+\Big(\frac{\Delta_{m_-}}{z-1}+\Delta_{m_+}-\frac{\Delta_+}{z}-h_{1,2}+\frac{\Lambda^2(z-1)}{z^2}\Big)\Psi=0.\tag{5.10}$$

• $N_f=3$: $\Psi(z,\Lambda)=\big\langle\Delta_{m_+}\big|\Phi_{\Delta_{m_-}}(1)\Phi_{1,2}(z)\big|\Delta_+,\Lambda,m\big\rangle$,

$$\begin{aligned}&b^2z(z-1)\Psi_{zz}-(2z-1)\Psi_z-\frac{\Lambda}{z}\Psi_\Lambda\\&+\Big(\frac{\Delta_{m_-}}{z-1}+\Delta_{m_+}-\frac{\Delta_+}{z}-h_{1,2}-\frac{2m\Lambda(z-1)}{z^2}-\frac{\Lambda^2(z-1)}{z^3}\Big)\Psi=0.\end{aligned}\tag{5.11}$$

• $N_f=4$: $\Psi(z,t)=\big\langle\Delta_{m_4}\big|\Phi_{\Delta_{m_3}}(1)\Phi_{1,2}(z)\Phi_{\Delta_{m_2}}(t)\big|\Delta_{m_1}\big\rangle$,

$$\begin{aligned}&b^2(z-1)z\Psi_{zz}-(2z-1)\Psi_z+\frac{(t-1)t}{(z-t)}\Psi_t\\&+\Big(\frac{\Delta_{m_3}}{z-1}+\Delta_{m_4}-\frac{\Delta_{m_1}}{z}-\frac{\Delta_{m_2}(t^2-2tz+z)}{(z-t)^2}-h_{1,2}\Big)\Psi=0.\end{aligned}\tag{5.12}$$

For the block of the form $\Psi(z)=\langle A|\mathcal{O}_1(1)\Phi_{1,2}(z)\cdots|B\rangle$ ($N_f=2^{(2)},3$ and 4), a convenient choice of the vector field ξ is $\xi=\frac{x(x-1)}{x-z}\frac{\partial}{\partial x}$.

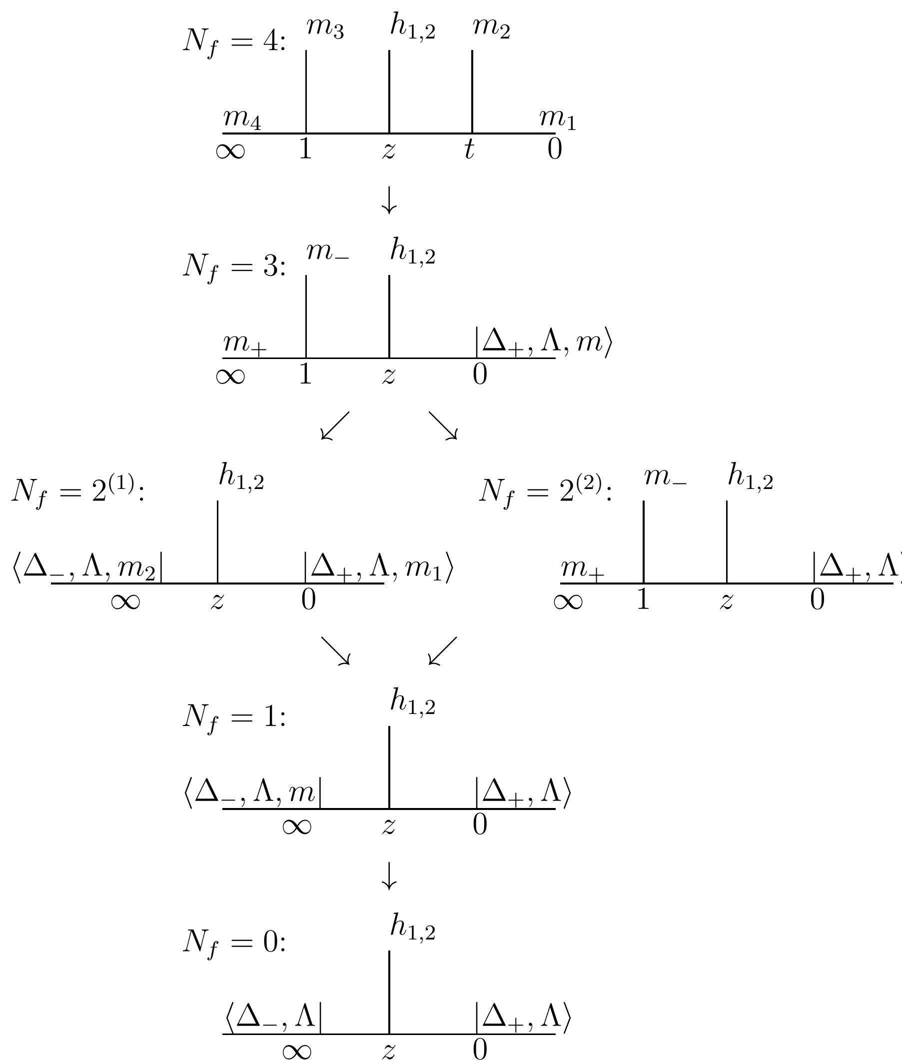

Figure 2: Conformal blocks and their degenerations

5.3 Quantum Seiberg-Witten curves

The CFT differential equation in previous subsection can be considered as a natural candidate for the speculated quantized Seiberg-Witten curve, that is an operator version of the equation $x^2 = \phi_2(z)$ (see the end of section 5 of [2]). By a gauge transformation[7] $\Psi = U\mathcal{Z}$, it can be written in the form $\mathcal{D}^{\rm SW}_{N_f}\mathcal{Z} = 0$ which looks like the Seiberg-Witten curve in the standard brane set-up. The operator $\widehat{v} = bz\partial_z$ is a quantization of the variable $v = zx$.

• $N_f = 0$: $U = z^{\Delta_- - \Delta_+ - h_{1,2}}$,

$$\mathcal{D}^{\rm SW}_0 = \frac{1}{4}\Lambda\partial_\Lambda + 2a\widehat{v} + \widehat{v}^2 + \Lambda^2(z + \frac{1}{z}). \tag{5.13}$$

• $N_f = 1$: $U = e^{\frac{\Lambda}{b}z} z^{\Delta_- - \Delta_+ - h_{1,2}}$,

$$\mathcal{D}^{\rm SW}_1 = \frac{1}{3}\Lambda\partial_\Lambda + (2a + \frac{1}{6b})\widehat{v} + \widehat{v}^2 + 2\Lambda z\Big(\widehat{v} + a + \frac{1}{4b} + \frac{b}{2} - m\Big) + \frac{\Lambda^2}{z}. \tag{5.14}$$

• $N_f = 2^{(1)}$: $U = e^{2\Lambda^2 + \frac{\Lambda}{b}(z + \frac{1}{z})} z^{\Delta_- - \Delta_+ - h_{1,2}}$,

$$\mathcal{D}^{\rm SW}_{2^{(1)}} = \frac{1}{2}\Lambda\partial_\Lambda + 2a\widehat{v} + \widehat{v}^2 - \frac{2\Lambda}{z}\Big(\widehat{v} + a - \frac{1}{4b} - \frac{b}{2} + m_1\Big) + 2\Lambda z\Big(\widehat{v} + a + \frac{1}{4b} + \frac{b}{2} - m_2\Big). \tag{5.15}$$

• $N_f = 2^{(2)}$: $U = (z-1)^{\frac{1+b^2-2bm_-}{2b^2}} z^{\Delta_- - \Delta_+ - h_{1,2}}$,

$$\mathcal{D}^{\rm SW}_{2^{(2)}} = \frac{1}{2}\Lambda\partial_\Lambda - \Lambda^2 + \Big(2a + \frac{1}{2b}\Big)\widehat{v} + \widehat{v}^2 + \frac{\Lambda^2}{z} - z\Big(\widehat{v} + a + \frac{1}{4b} + \frac{b}{2} - m_- - m_+\Big)\Big(\widehat{v} + a + \frac{1}{4b} + \frac{b}{2} - m_- + m_+\Big). \tag{5.16}$$

• $N_f = 3$: $U = e^{-\Lambda(b + \frac{1}{b} - 2m_- - \frac{1}{bz})}(z-1)^{\frac{1+b^2-2bm_-}{2b^2}} z^{\Delta_- - \Delta_+ - h_{1,2}}$,

$$\begin{aligned}\mathcal{D}^{\rm SW}_3 &= \Lambda\partial_\Lambda + (2a - \frac{1}{2b} - b + 2m)\Lambda + (2a + \frac{1}{2b} + 2\Lambda)\widehat{v} + \widehat{v}^2 - \frac{2\Lambda}{z}\Big(\widehat{v} + a - \frac{1}{4b} - \frac{b}{2} + m\Big)\\ &-z\Big(\widehat{v} + a + \frac{1}{4b} + \frac{b}{2} - m_- - m_+\Big)\Big(\widehat{v} + a + \frac{1}{4b} + \frac{b}{2} - m_- + m_+\Big).\end{aligned} \tag{5.17}$$

• $N_f = 4$: $U = (1-t)^{-\frac{(1+b^2-2bm_2)(1+b^2-2bm_3)}{2b^2}}(1-\frac{t}{z})^{\frac{1+b^2-2bm_2}{2b^2}}(1-z)^{\frac{1+b^2-2bm_3}{2b^2}} z^{\Delta_- - \Delta_+ - h_{1,2}} t^{\Delta_+ - \Delta_{m_1} - \Delta_{m_2}}$,

$$\begin{aligned}\mathcal{D}^{\rm SW}_4 &= (1-t)t\partial_t + (\mu_1\mu_2 + \mu_3\mu_4)t + \Big((2a + \frac{1}{2b})(1-t) + (\mu_1 + \mu_2 + \mu_3 + \mu_4)t\Big)\widehat{v}\\ &+(1+t)\widehat{v}^2 - \frac{t}{z}(\widehat{v} + \mu_1)(\widehat{v} + \mu_2) - z(\widehat{v} + \mu_3)(\widehat{v} + \mu_4),\end{aligned} \tag{5.18}$$

[7] The factors U are determined by comparison with the gauge theory (localization) results. It may be interesting to note that the factor U for $N_f = 4$ case is exactly the same as the pre-factor appearing in the integral (free field) representation of the conformal block.

where

$$\begin{aligned} \mu_1 &= a - \tfrac{1}{4b} - \tfrac{b}{2} - m_1 + m_2, \quad \mu_2 = a - \tfrac{1}{4b} - \tfrac{b}{2} + m_1 + m_2, \\ \mu_3 &= a + \tfrac{1}{4b} + \tfrac{b}{2} - m_3 - m_4, \quad \mu_4 = a + \tfrac{1}{4b} + \tfrac{b}{2} - m_3 + m_4. \end{aligned} \tag{5.19}$$

We remark that the equations for the (irregular) conformal blocks considered here are the same as the Schrödinger equation for quantum Painlevé equations [49]. The connection between CFT, iso-monodromy deformation, and Seiberg-Witten curves are natural because (i) CFT (KZ equation for example) are the quantization of iso-monodromy deformation (Schlesinger system) [50, 51] and (ii) the cubic equations which determine the classical Painlevé Hamiltonians coincide with the $SU(2)$ Seiberg-Witten curves [52].

5.4 Degenerations

Here, we give the degeneration scheme that connect the N_f and $N_f - 1$ equations. We use the notation Λ_{N_f} and z_{N_f} for Λ, z variables of $N_f = 0, 1, 2^{(1)}, 2^{(2)}, 3$ and 4.

- We have $\mathcal{D}_4^{\mathrm{SW}} \to \mathcal{D}_3^{\mathrm{SW}}$ under the limit

$$m_1 + m_2 = m, \quad m_3 = m_-, \quad m_4 = m_+, \quad tm_2 = \Lambda_3, \quad m_2 \to \infty, \quad z_4 = z_3. \tag{5.20}$$

- We have $\mathcal{D}_3^{\mathrm{SW}} \to \mathcal{D}_{2^{(1)}}^{\mathrm{SW}}$ under the limit

$$m = m_1, \quad m_+ + m_- = m_2, \quad m_- \to \infty, \quad \Lambda_3 m_- = \Lambda_{2^{(1)}}^2, \quad \frac{z_3}{\Lambda_3} = \frac{z_{2^{(1)}}}{\Lambda_{2^{(1)}}}. \tag{5.21}$$

- We have $\mathcal{D}_3^{\mathrm{SW}} \to \mathcal{D}_{2^{(2)}}^{\mathrm{SW}}$ under the limit

$$m \to \infty, \quad -2\Lambda_3 m = \Lambda_{2^{(2)}}^2, \quad z_3 = z_{2^{(2)}}. \tag{5.22}$$

- We have $\mathcal{D}_{2^{(1)}}^{\mathrm{SW}} \to \mathcal{D}_1^{\mathrm{SW}}$ under the limit

$$m_1 \to \infty, \quad m_2 = m, \quad -2\Lambda_{2^{(1)}}^2 m_1 = \Lambda_1^3, \quad z_{2^{(1)}}\Lambda_{2^{(1)}} = z_1\Lambda_1. \tag{5.23}$$

- We have $\mathcal{D}_{2^{(2)}}^{\mathrm{SW}} \to \mathcal{D}_1^{\mathrm{SW}}$ under the limit

$$m_- \to \infty, \quad m_+ + m_- = m, \quad \Lambda_{2^{(2)}}^2 m_- = \Lambda_1^3, \quad \frac{z_{2^{(2)}}}{\Lambda_{2^{(2)}}^2} = \frac{z_1}{\Lambda_1^2}. \tag{5.24}$$

- We have $\mathcal{D}_1^{\mathrm{SW}} \to \mathcal{D}_0^{\mathrm{SW}}$ under the limit

$$m \to \infty, \quad -2\Lambda_1^3 m = \Lambda_0^4, \quad \frac{z_1}{\Lambda_1^2} = \frac{z_0}{\Lambda_0^2}. \tag{5.25}$$

In all the cases, the degenerations of the CFT are consistent with the decoupling limit of the gauge theory as discussed in the case of without surface operator [14]. By the parameter relation (5.33), the decoupling limit are described as follows:

$$\begin{array}{lll} 4 \to 3 & : & t = \Lambda_3 \frac{\hbar}{M_2}, \quad M_2 \to \infty, \\ 3 \to 2^{(1)} & : & \Lambda_3 = \Lambda_{2^{(1)}}^2 \frac{\hbar}{-M_3}, \quad z_3 = z_{2^{(1)}} \Lambda_{2^{(1)}} \frac{\hbar}{-M_3}, \quad -M_3 \to \infty, \\ 3 \to 2^{(2)} & : & \Lambda_3 = \Lambda_{2^{(2)}}^2 \frac{\hbar}{4M_4}, \quad M_4 \to \infty, \\ 2^{(1)} \to 1 & : & \Lambda_{2^{(1)}}^2 = \Lambda_1^3 \frac{\hbar}{4M_4}, \quad z_{2^{(1)}} = z_1 (\Lambda_1 \frac{\hbar}{4M_4})^{-1/2}, \quad M_4 \to \infty, \\ 2^{(2)} \to 1 & : & \Lambda_{2^{(2)}}^2 = \Lambda_1^3 \frac{\hbar}{-M_3}, \quad z_{2^{(2)}} = z_1 \Lambda_1 \frac{\hbar}{-M_3}, \quad -M_3 \to \infty \\ 1 \to 0 & : & \Lambda_1^3 = \Lambda_0^4 \frac{\hbar}{-4M_1}, \quad z_1 = z_0 (\Lambda_0 \frac{\hbar}{-4M_1})^{2/3}, \quad -M_1 \to \infty, \\ & & \hbar = (\epsilon_1 \epsilon_2)^{1/2}. \end{array} \tag{5.26}$$

The degeneration relation of $N_f = 4 \to N_f = 3$ (and similarly $N_f = 3 \to N_f = 2^{(1)}$) has simple interpretation in operator level as follows. Consider the product

$$\big|\psi\big\rangle = z^{\Delta_{m_1} + \Delta_{m_2} - \Delta_a} \Phi_{\Delta_{m_1}}(z) \big|\Delta_{m_2}\big\rangle. \tag{5.27}$$

By the definition of the primary filed, it satisfies

$$L_n \big|\psi\big\rangle = z^n \big(z\partial_z + \Delta_a + n\Delta_{m_1} - \Delta_{m_2} \big) \big|\psi\big\rangle. \quad (n > 0) \tag{5.28}$$

Here, we will put

$$z = \epsilon\Lambda, \quad m_1 = -\frac{1}{\epsilon} - m, \quad m_2 = \frac{1}{\epsilon}. \tag{5.29}$$

Then, under the limit $\epsilon \to 0$, we have $\Delta_a + n\Delta_{m_1} - \Delta_{m_2} = \frac{1-n}{\epsilon^2} + \frac{-2mn}{\epsilon} + O(\epsilon^0)$ and

$$L_1 \big|\psi\big\rangle = -2m\Lambda \big|\psi\big\rangle, \quad L_2 \big|\psi\big\rangle = -\Lambda^2 \big|\psi\big\rangle, \quad L_n \big|\psi\big\rangle = 0, \quad (n \geq 3). \tag{5.30}$$

Thus, the Gaiotto state $\big|\Delta_a, \Lambda, m\big\rangle$ can be obtained as a degeneration limit (5.29) of two primaries.

5.5 Solutions

The equation $\mathcal{D}_4^{\mathrm{SW}} \mathcal{Z}(z,t) = 0$ has the following series solution

$$\mathcal{Z}(z,t) = \sum_{n=0}^{\infty} \mathcal{Z}_n(z) t^n, \quad \mathcal{Z}_n(z) = \sum_{k=-n}^{\infty} c_{n,k} z^k. \tag{5.31}$$

First few terms are as follows

$$
\begin{aligned}
\mathcal{Z}_0 &= 1+\frac{2\mu_3\mu_4}{2b^2+4ab+1}z+\frac{2\mu_3(b+\mu_3)\mu_4(b+\mu_4)}{(2b^2+4ab+1)\,(4b^2+4ab+1)}z^2+\mathcal{O}(z^3),\\
\mathcal{Z}_1 &= \frac{2\mu_1\mu_2}{2b^2-4ab+1}\frac{1}{z}-\mu_1\mu_2-\mu_3\mu_4+\frac{2\mu_1\mu_2(b-\mu_3)(b-\mu_4)}{2b^2-4ab+1}+\frac{2(b+\mu_1)(b+\mu_2)\mu_3\mu_4}{2b^2+4ab+1}+\mathcal{O}(z),\\
\mathcal{Z}_2 &= \frac{2(b-\mu_1)\mu_1(b-\mu_2)\mu_2}{(2b^2-4ab+1)\,(4b^2-4ab+1)}\frac{1}{z^2}+\mathcal{O}(\frac{1}{z}),
\end{aligned}
\tag{5.32}
$$

These agree with the localization results (3.28) by the following correspondence[8] :

$$
\begin{aligned}
&a\to\frac{-a-\epsilon_2/4}{\sqrt{\epsilon_1\epsilon_2}},\quad b\to\sqrt{\frac{\epsilon_1}{\epsilon_2}},\quad z\to -x,\quad \frac{t}{z}\to -y,\\
&\mu_1\to\frac{-a-\epsilon_1-\epsilon_2+2M_2}{\sqrt{\epsilon_1\epsilon_2}},\quad \mu_2\to\frac{-a-2M_4}{\sqrt{\epsilon_1\epsilon_2}},\quad \mu_3\to\frac{-a+\epsilon_1-2M_1}{\sqrt{\epsilon_1\epsilon_2}},\quad \mu_4\to\frac{-a+2M_3}{\sqrt{\epsilon_1\epsilon_2}}.
\end{aligned}
\tag{5.33}
$$

The coefficients of the border terms z^n and $(\frac{t}{z})^n$ are always factorized. From the CFT point of view, this can be understood by fusion (not degeneration) of primary operators. More precisely, for $t\to 0$, then $\Phi_{\Delta(m_2)}(t)$ and $|\Delta(m_1)\rangle$ are fused and we have

$$
\mathcal{Z}(z,0)={}_2F_1(\frac{\mu_3}{b},\frac{\mu_4}{b},\frac{2a}{b}+1+\frac{1}{2b^2},z). \tag{5.34}
$$

Similarly, for $z\to 0$ (with $u=\frac{t}{z}$ fixed), then $\langle\Delta(m_4)|$ and $\Phi_{\Delta(m_2)}(1)$ are fused and

$$
\mathcal{Z}(z,uz)={}_2F_1(-\frac{\mu_1}{b},-\frac{\mu_2}{b},-\frac{2a}{b}+1+\frac{1}{2b^2},u). \tag{5.35}
$$

For the degenerate cases $N_f\le 3$, one can also solve the differential equations $\mathcal{D}^{\mathrm{SW}}_{N_f}\mathcal{Z}=0$ in series expansion. Alternatively, such solutions can be obtained through the limiting procedure starting form $N_f=4$ case. Since the limit can be taken term by term with respect to the variables z and Λ (or t), we will illustrate the procedure on simplest examples.

[8]We have checked this up to the order 7 in x and y variables.

The first example is the term z in $N_f = 4$ solution and its degenerations:

$$\begin{tikzcd}
& \frac{\left(2b^2+4ab-4m_3b-4m_4b+1\right)\left(2b^2+4ab-4m_3b+4m_4b+1\right)z}{8b^2(2b^2+4ab+1)} \arrow[d] & \\
& \frac{\left(2b^2+4ab-4m_-b-4m_+b+1\right)\left(2b^2+4ab-4m_-b+4m_+b+1\right)z}{8b^2(2b^2+4ab+1)} \arrow[dl] \arrow[dr] & \\
-\frac{\Lambda\left(2b^2+4ab-4m_2b+1\right)z}{b(2b^2+4ab+1)} \arrow[dr] & & \frac{\left(2b^2+4ab-4m_-b-4m_+b+1\right)\left(2b^2+4ab-4m_-b+4m_+b+1\right)z}{8b^2(2b^2+4ab+1)} \arrow[dl] \\
& -\frac{\Lambda\left(2b^2+4ab-4mb+1\right)z}{b(2b^2+4ab+1)} \arrow[d] & \\
& -\frac{2\Lambda^2 z}{2b^2+4ab+1}. &
\end{tikzcd}$$

Here, the arrows are the same meaning as Fig.2. This degeneration corresponds to the decoupling of the fundamental matter attached to the vertical brane at $z = 0$. The next example represents the similar decoupling process around the brane at $z = \infty$:

$$\begin{tikzcd}
& \frac{\left(2b^2-4ab-4m_1b-4m_2b+1\right)\left(2b^2-4ab+4m_1b-4m_2b+1\right)t}{8b^2(2b^2-4ab+1)z} \arrow[d] & \\
& -\frac{\left(2b^2-4ab-4mb+1\right)\Lambda}{b(2b^2-4ab+1)z} \arrow[dl] \arrow[dr] & \\
-\frac{\Lambda\left(2b^2-4ab-4m_1b+1\right)}{b(2b^2-4ab+1)z} \arrow[dr] & & \frac{2\Lambda^2}{(-2b^2+4ab-1)z} \arrow[dl] \\
& \frac{2\Lambda^2}{(-2b^2+4ab-1)z} \arrow[d] & \\
& \frac{2\Lambda^2}{(-2b^2+4ab-1)z}. &
\end{tikzcd}$$

6 B model computations via Seiberg-Witten curve

In [2] it is argued that the Seiberg-Witten curve arises in the "semiclassical limit" $\epsilon_{1,2} \ll a_i, m_i$ of the expectation value of the energy momentum tensor. For example, by taking the limit $\hbar \to 0$ one finds the following Seiberg-Witten curves $x^2 = \phi_n^{\rm SW}(z)$ [13],

$$
\begin{aligned}
&SU(2), N_f = 0: \ \langle \Delta_a, \Lambda | T(z) | \Delta_a, \Lambda \rangle \ \longrightarrow \ -\frac{1}{\hbar^2}\phi_0^{\rm SW}(z) \langle \Delta_a, \Lambda | \Delta_a, \Lambda \rangle \\
&\quad \phi_0^{\rm SW}(z) = M_0(z)^2 \sigma_0(z), \ \sigma_0(z) := -z\big(z^2 - \frac{u}{\Lambda^2} z + 1\big), \ M_0(z) := \frac{\Lambda}{z^2}, \qquad (6.1)\\
&SU(2), N_f = 1: \ \langle \Delta_a, \Lambda, m | T(z) | \Delta_a, \Lambda \rangle \ \longrightarrow \ -\frac{1}{\hbar^2}\phi_1^{\rm SW}(z) \langle \Delta_a, \Lambda, m | \Delta_a, \Lambda \rangle \\
&\quad \phi_1^{\rm SW}(z) = M_1(z)^2 \sigma_1(z), \ \sigma_1(z) := z\big(z^3 + \frac{2m}{\Lambda} z^2 + \frac{u}{\Lambda^2} z - 1\big), \ M_1(z) := \frac{\Lambda}{z^2}, \qquad (6.2)\\
&SU(2), N_f = 2: \ \langle \Delta_a, \Lambda, m_2 | T(z) | \Delta_a, \Lambda, m_1 \rangle \ \longrightarrow \ -\frac{1}{\hbar^2}\phi_2^{\rm SW}(z) \langle \Delta_a, \Lambda, m_2 | \Delta_a, \Lambda, m_1 \rangle \\
&\quad \phi_2^{\rm SW}(z) = M_2(z)^2 \sigma_2(z), \ \sigma_2(z) := z^4 + \frac{2m_2}{\Lambda} z^3 + \frac{u}{\Lambda^2} z^2 + \frac{2m_1}{\Lambda} z + 1, \ M_2(z) := \frac{\Lambda}{z^2}, \ (6.3)
\end{aligned}
$$

where in this section the subscript n of $\phi_n^{\rm SW}(z) = M_n(z)^2 \sigma_n(z)$ stands for the number of flavors. The Coulomb moduli parameter $u = a^2 + \mathcal{O}(\Lambda)$ in each $\mathcal{N} = 2$ supersymmetric gauge theory is determined from the period

$$
a(u) = \oint_A \lambda_{\rm SW}(z), \quad \lambda_{\rm SW}(z) := x(z) dz, \tag{6.4}
$$

where A is the A-cycle on the Seiberg-Witten curve $x^2 = \phi_n^{\rm SW}(z)$. Using the discussion in [3], one can find that the leading term (disk amplitude) F_{-1} of the free energy $F_{\rm null}$ in section 4 is related to the Seiberg-Witten curve by

$$
\left(\frac{\partial F_{-1}(z)}{\partial z} \right)^2 = \phi_n^{\rm SW}(z), \quad \longrightarrow \quad F_{-1}(z) = \pm \int^z \lambda_{\rm SW}(z'). \tag{6.5}
$$

Note that in this computation we do not know how to determine the constant of integration for $F_{-1}(z)$. Actually for (6.1) – (6.3), we can check that the right hand side of (6.5) agrees with the computations (4.17), (4.31) and (4.44) in section 4 for the first few orders in Λ except constant terms in the insertion point z of the degenerate operator.

In [29], Eynard and Orantin defined the free energies on arbitrary complex plane curves by the topological recursion which has its origin to the loop equation in matrix models. In [6], it was claimed that the correlation functions in the CFT for $\mathcal{N} = 2$ superconformal quiver gauge theories can be related to the free energies defined by the

topological recursion on the Seiberg-Witten curves obtained from the energy momentum tensor of the CFT as (6.1) – (6.3). In this section we generalize their claim to asymptotically free theories. Following the construction of Eynard and Orantin, let us define the free energies $\mathcal{F}^{(g,h)}_{\mathrm{SW}}(z_1,\ldots,z_h)$, $g,h\in\mathbb{Z}_{\geq 0}$, $h\geq 1$ on the Seiberg-Witten curve $\mathcal{C}_{\mathrm{SW}}$: $x^2=\phi_n^{\mathrm{SW}}(z)=M_n(z)^2\sigma_n(z)$ by

$$\begin{aligned}
&\mathcal{F}^{(g,h)}_{\mathrm{SW}}(z_1,\ldots,z_h):=\int^{z_1}\cdots\int^{z_h}W^{(g,h)}(z_1',\ldots,z_h'),\\
&W^{(0,1)}(z):=\lambda_{\mathrm{SW}}(z),\quad W^{(0,2)}(z_1,z_2):=B(z_1,z_2)-\frac{dz_1dz_2}{(z_1-z_2)^2},\\
&W^{(g,h)}(z_1,\ldots,z_h):=\widetilde{W}^{(g,h)}(z_1,\ldots,z_h)\quad\text{for}\quad(g,h)\neq(0,1),(0,2),
\end{aligned}\tag{6.6}$$

where $\mathcal{F}^{(0,1)}_{\mathrm{SW}}(z)$ is nothing but the disk amplitude (6.5). The multilinear meromorphic differentials $W^{(g,h)}(z_1,\ldots,z_h)$ are defined on $\mathcal{C}_{\mathrm{SW}}$ by the topological recursion relation

$$\begin{aligned}
&\widetilde{W}^{(0,1)}(z):=0,\quad \widetilde{W}^{(0,2)}(z_1,z_2):=B(z_1,z_2),\\
&dE_{q,\bar{q}}(z):=\frac{1}{2}\int_q^{\bar{q}}B(z,\xi),\ \text{near a branch point }q_i,\\
&\widetilde{W}^{(g,h+1)}(z,z_1,\ldots,z_h):=\sum_{q_i\in\mathcal{C}_{\mathrm{SW}}}\operatorname*{Res}_{q=q_i}\frac{dE_{q,\bar{q}}(z)}{\lambda_{\mathrm{SW}}(q)-\lambda_{\mathrm{SW}}(\bar{q})}\Bigg\{\widetilde{W}^{(g-1,h+2)}(q,\bar{q},z_1,\ldots,z_h)\\
&\qquad\qquad+\sum_{\ell=0}^{g}\sum_{J\subset H}\widetilde{W}^{(g-\ell,|J|+1)}(q,z_J)\widetilde{W}^{(\ell,|H|-|J|+1)}(\bar{q},z_{H\backslash J})\Bigg\},
\end{aligned}\tag{6.7}$$

where q_i are the branch points on $\mathcal{C}_{\mathrm{SW}}$, $H=\{1,\ldots,h\}$, $J=\{i_1,\ldots,i_j\}\subset H$ and $z_J=\{z_{i_1},\ldots,z_{i_j}\}$. q and $\bar{q}$ denote the positions on the upper and the lower sheet, respectively. The Bergman kernel $B(z_1,z_2)$ is given by the Akemann's formula [53, 54],

$$B(z_1,z_2)=\frac{dz_1dz_2}{2(z_1-z_2)^2}\left(\frac{2f(z_1,z_2)+G(k)(z_1-z_2)^2}{2\sqrt{\sigma_n(z_1)\sigma_n(z_2)}}+1\right),\tag{6.8}$$

$$f(z_1,z_2):=z_1^2z_2^2+\frac{1}{2}z_1z_2(z_1+z_2)S_1+\frac{1}{6}(z_1^2+4z_1z_2+z_2^2)S_2+\frac{1}{2}(z_1+z_2)S_3+S_4,\tag{6.9}$$

$$G(k):=-\frac{1}{3}S_2+(q_1q_2+q_3q_4)-\frac{E(k)}{K(k)}(q_1-q_3)(q_2-q_4),\tag{6.10}$$

$$K(k)=\int_0^1\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},\ E(k)=\int_0^1dt\sqrt{\frac{1-k^2t^2}{1-t^2}},\ k^2=\frac{(q_1-q_2)(q_3-q_4)}{(q_1-q_3)(q_2-q_4)},\tag{6.11}$$

where S_k is the coefficient of z^{4-k} in $\sigma_n(z)$. $K(k)$ (resp. $E(k)$) is the complete elliptic integral of the first (resp. the second) kind with the modulus k.

For $\mathcal{N}=2$ superconformal quiver gauge theories with the self-dual constraint $\epsilon_1 = -\epsilon_2 = i\hbar$, the claim of [6] may be summarized as

$$F^{(k)}_{\mathrm{CFT}}(z_1,\ldots,z_{\widetilde{h}}) = \sum_{2-2g-h=-k} \frac{1}{h!} \sum_{i_1,\ldots,i_h=1}^{\widetilde{h}} \mathcal{F}^{(g,h)}_{\mathrm{SW}}(z_{i_1},\ldots,z_{i_h}), \tag{6.12}$$

where the left hand side is the $\widetilde{h}$-points free energy defined by (4.46) on the CFT side with internal channels chosen so that the result is symmetric in variables $z_1,\ldots,z_{\widetilde{h}}$. Under this constraint on the internal channel, the left hand side is essentially fixed. On the other hand, there exist ambiguities of the constants of integration in (6.6). Thus we will make a more modest proposal by keeping only universal terms on the right hand side of (6.12), which are independent of these ambiguities. For both the superconformal and the asymptotically free theories we expect that at least a part of the relation (6.12) is valid,

$$F^{(h-2)}_{\mathrm{CFT}}(z_1,\ldots,z_h) = \mathcal{F}^{(0,h)}_{\mathrm{SW}}(z_1,\ldots,z_h), \tag{6.13}$$

where $\mathcal{F}^{(0,h)}_{\mathrm{SW}}(z_1,\ldots,z_h)$ is the summation of all the universal terms which are of the form $a_{n_1 n_2 \ldots n_h} z_1^{n_1} z_2^{n_2} \cdots z_h^{n_h}$ $(n_i \in \mathbb{Z})$ with the condition $\prod_{i=1}^h n_i \neq 0$. In the rest of this section we will explicitly check the relation (6.13) for $N_f = 0$ and $N_f = 1$.

6.1 Pure $SU(2)$

Here we compute the free energies on the Seiberg-Witten curve (6.1) corresponding to pure $SU(2)$ supersymmetric gauge theory. The period (6.4) is obtained from the complete elliptic integral as follows:

$$\frac{da(u)}{du} = \oint_A \frac{\partial \lambda_{\mathrm{SW}}(z)}{\partial u} = \frac{1}{2\Lambda} \oint_A \frac{dz}{\sqrt{\sigma_0(z)}} = \frac{1}{\pi \Lambda \sqrt{q_3 - q_1}} K(k), \quad k^2 = \frac{q_1 - q_2}{q_1 - q_3}, \tag{6.14}$$

where $q_1 = 0$, $q_2 = (u - \sqrt{u^2 - 4\Lambda^4})/2\Lambda^2$ and $q_3 = (u + \sqrt{u^2 - 4\Lambda^4})/2\Lambda^2$ are the branch points of the curve (6.1). Thus one obtains

$$u(a) = a^2 + \frac{\Lambda^4}{2a^2} + \frac{5\Lambda^8}{32a^6} + \frac{9\Lambda^{12}}{64a^{10}} + \frac{1469\Lambda^{16}}{8192a^{14}} + \frac{4471\Lambda^{20}}{16384a^{18}} + \mathcal{O}(\Lambda^{24}). \tag{6.15}$$

To compute the annulus amplitude $\mathcal{F}^{(0,2)}_{\mathrm{SW}}(z_1,z_2)$ on the curve (6.1), taking the limit $q_4 \to \infty$ in (6.8), one obtains the Bergman kernel on the curve by replacing $f(z_1,z_2)$ and

$G(k)$ with

$$\widetilde{f}(z_1,z_2) = -\frac{1}{2}z_1z_2(z_1+z_2) - \frac{1}{6}(z_1^2+4z_1z_2+z_2^2)\widetilde{S}_1 - \frac{1}{2}(z_1+z_2)\widetilde{S}_2 - \widetilde{S}_3,$$
$$\widetilde{G}(k) = \frac{1}{3}\widetilde{S}_1 + q_3 + \frac{E(k)}{K(k)}(q_1-q_3), \tag{6.16}$$

where $\widetilde{S}_k$ is the coefficient of z^{3-k} in $\sigma_0(z)$. Thus we obtain the annulus amplitude

$$\begin{aligned}\mathcal{F}_{\rm SW}^{(0,2)}(z_1,z_2) &= \frac{z_1^2z_2^2+1}{16a^4z_1z_2}\Lambda^4 + \frac{(z_1+z_2)(z_1^3z_2^3+1)}{32a^6z_1^2z_2^2}\Lambda^6\\ &+\frac{10(z_1^2+z_2^2)(z_1^4z_2^4+1)+9z_1z_2(z_1^4z_2^4+1)+32z_1^2z_2^2(z_1^2z_2^2+1)-4z_1^2z_2^2(z_1^2+z_2^2)}{512a^8z_1^3z_2^3}\Lambda^8\\ &+\mathcal{O}(\Lambda^{10}).\end{aligned} \tag{6.17}$$

We can see that the amplitude agrees with (B.7) up to Λ^8. Hence the relation (6.13) is correct as was expected.

Higher topology amplitudes are iteratively computed by the recursion (6.7) and the multilinear meromorphic differentials $W^{(g,h)}(z_1,\ldots,z_h)$ can be expanded by the kernel differentials [28],

$$\chi_i^{(n)}(z) := \operatorname*{Res}_{q=q_i}\left(-\frac{2dE_{q,\bar{q}}(z)}{\lambda_{\rm SW}(q)-\lambda_{\rm SW}(\bar{q})}\frac{dq}{(q-q_i)^n}\right). \tag{6.18}$$

For example, $W^{(0,3)}(z_1,z_2,z_3)$ is written as

$$\begin{aligned}W^{(0,3)}(z_1,z_2,z_3) &= \sum_{q_i}\operatorname*{Res}_{q=q_i}\frac{2dE_{q,\bar{q}}(z)}{\lambda_{\rm SW}(q)-\lambda_{\rm SW}(\bar{q})}B(z_2,q)B(z_3,\bar{q})\\ &= \frac{1}{2}\sum_{q_i}M_n(q_i)^2\sigma_n'(q_i)\chi_i^{(1)}(z_1)\chi_i^{(1)}(z_2)\chi_i^{(1)}(z_3),\end{aligned} \tag{6.19}$$

$$\chi_i^{(1)}(z) = \frac{dz}{2M_n(q_i)\sigma_n'(q_i)\sqrt{\sigma_n(z)}}\left(G(k)+\frac{2f(z,q_i)}{(z-q_i)^2}\right). \tag{6.20}$$

Thus we obtain the three-holed amplitude $\mathcal{F}_{\rm SW}^{(0,3)}(z_1,z_2,z_3)$ on the Seiberg-Witten curve (6.1),

$$\begin{aligned}\mathcal{F}_{\rm SW}^{(0,3)}(z_1,z_2,z_3) &= \frac{z_1^2z_2^2z_3^2-1}{16a^7z_1z_2z_3}\Lambda^6 + \frac{3\big(z_1^3z_2^3z_3^3(z_1+z_2+z_3)-(z_1z_2+z_2z_3+z_3z_1)\big)}{64a^9z_1^2z_2^2z_3^2}\Lambda^8\\ &+\bigg\{\frac{z_1^2+z_2^2+z_3^2-(z_1^2z_2^2+z_2^2z_3^2+z_3^2z_1^2)}{128a^{11}z_1z_2z_3}+\frac{5\big(z_1^4z_2^4z_3^4(z_1^2+z_2^2+z_3^2)-(z_1^2z_2^2+z_2^2z_3^2+z_3^2z_1^2)\big)}{128a^{11}z_1^3z_2^3z_3^3}\\ &+\frac{9\big(z_1^3z_2^3z_3^3(z_1z_2+z_2z_3+z_3z_1)-(z_1+z_2+z_3)\big)}{256a^{11}z_1^2z_2^2z_3^2}+\frac{9(z_1^2z_2^2z_3^2-1)}{64a^{11}z_1z_2z_3}\bigg\}\Lambda^{10}+\mathcal{O}(\Lambda^{12}),\end{aligned} \tag{6.21}$$

and in (B.11) we checked the relation (6.13) up to Λ^6.

6.2 $SU(2)$ with one fundamental matter

We compute the annulus amplitude $\mathcal{F}^{(0,2)}_{\rm SW}(z_1,z_2)$ on the Seiberg-Witten curve (6.2) corresponding to $SU(2)$ supersymmetric gauge theory with one fundamental matter. The period (6.4) is computed from

$$\frac{da(u)}{du}=\frac{1}{\pi\Lambda\sqrt{(q_1-q_3)(q_2-q_4)}}K(k),\quad k^2=\frac{(q_1-q_2)(q_3-q_4)}{(q_1-q_3)(q_2-q_4)},\tag{6.22}$$

where $q_{1,2}=-(m\pm\sqrt{m^2-u})/\Lambda+\Lambda^2/(2(m^2\pm m\sqrt{m^2-u}-u))+\mathcal{O}(\Lambda^5), q_3=0$ and $q_4=\Lambda^2/u+\mathcal{O}(\Lambda^5)$. One finds

$$u(a)=a^2-\frac{m\Lambda^3}{a^2}-\frac{(3a^2-5m^2)\Lambda^6}{8a^6}+\mathcal{O}(\Lambda^9).\tag{6.23}$$

Then, from (6.8) we obtain the annulus amplitude

$$\begin{aligned}\mathcal{F}^{(0,2)}_{\rm SW}(z_1,z_2)=&-\frac{(a^2-m^2)z_1z_2}{4a^4}\Lambda^2+\frac{m(a^2-m^2)z_1z_2(z_1+z_2)}{4a^6z_1z_2}\Lambda^3\\&+\frac{2(a^2-m^2)(a^2-5m^2)(z_1^2+z_2^2)z_1^2z_2^2+(a^2-m^2)(a^2-9m^2)z_1^3z_2^3+2a^4}{32a^8z_1z_2}\Lambda^4+\mathcal{O}(\Lambda^5).\end{aligned}\tag{6.24}$$

As before this agrees with $F^{(0)}_{\rm CFT}(z_1,z_2)$ up to Λ^4 (see (B.16)). Hence the relation (6.13) also holds in this case.

7 Geometric engineering and open topological B model

Hereafter we consider the open topological string on toric Calabi-Yau threefolds (local A model) which is expected to realize a surface operator in $\mathcal{N}=2$ $SU(2)$ gauge theories in four dimensions. In this section we compute the topological open string amplitude by combining the local mirror symmetry with the conjecture of remodeling the B model [27, 28], by which we have the equality between the local A model amplitudes and the free energies $\mathcal{F}^{(g,h)}(x_1,\ldots,x_h)$, $g,h\in\mathbb{Z}_{\geq 0},\ h\geq 1$ on the mirror curve

$$\mathcal{C}=\{x,y\in\mathbb{C}^*\mid H(x,y)=0\}\quad\subset\mathbb{C}^*\times\mathbb{C}^*,\tag{7.1}$$

computed by the topological recursion relation of Eynard and Orantin we employed in section 5. The free energies are defined by (6.6) and (6.7) under the replacement

$$\lambda_{\rm SW}(z),\quad\longrightarrow\quad\omega(x):=\log y(x)\frac{dx}{x}.\tag{7.2}$$

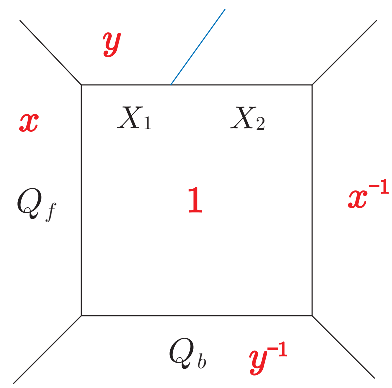

Figure 3: Local Hirzebruch surface $K_{\mathbf{F}_0}$

7.1 Toric brane on local Hirzebruch surface $K_{\mathbf{F}_0}$: pure $SU(2)$

The pure $SU(2)$ gauge theory is realized by the Hirzebruch surface $\mathbf{F}_0 = \mathbf{P}^1_f \times \mathbf{P}^1_b$, and we insert a toric brane on the base $\mathbf{P}^1_b$ as the blue line in Fig.3. The Kähler parameters Q_f, Q_b of $\mathbf{P}^1_f$, $\mathbf{P}^1_b$ and the parameters on the gauge theory side are related by [18],

$$Q_f = e^{-2\beta a}, \quad Q_b = X_1 X_2 = \beta^4 \Lambda^4, \quad X_1 := \beta^2 \Lambda^2 w^{-1}, \quad X_2 := \beta^2 \Lambda^2 w, \tag{7.3}$$

where β is a scale parameter which corresponds to the radius of the fifth dimension in the gauge theory and $X_1, (X_2)$ represents the distance between the toric brane and the trivalent vertex in the web diagram as indicated in Fig.3. The charge vectors of $K_{\mathbf{F}_0}$ are given by

$$\ell_b = (-2, 1, 0, 1, 0), \quad \ell_f = (-2, 0, 1, 0, 1). \tag{7.4}$$

By taking the local coordinate patch as Fig.3, the mirror curve which describes the moduli of the toric brane is obtained as

$$xy^2 + (x^2 + x + z_b)y + z_f x = 0, \quad \sigma(x) := (x^2 + x + z_b)^2 - 4z_f x^2, \tag{7.5}$$

where z_f, z_b are the moduli parameters of complex structure of the mirror Calabi-Yau threefold. The closed and open mirror maps are given by [55, 56, 57],

$$\begin{aligned} \log Q_b &= \log z_b + 2 \sum_{m,n\geq 0,(m,n)\neq(0,0)}^{\infty} \frac{(2m+2n-1)!}{m!^2 n!^2} z_b^m z_f^n, \\ \log \frac{Q_f}{Q_b} &= \log \frac{z_f}{z_b}, \quad X = \left(\frac{Q_b}{z_b}\right)^{\frac{1}{2}} x, \end{aligned} \tag{7.6}$$

where $X = X(x)$ is the open string moduli on the A model side. The disk amplitude is computed in a similar manner to [56],

$$\begin{aligned}
\mathcal{F}^{(0,1)}(Q_f, \Lambda, w) &= \int^x \omega(x') = \int^x \log\left\{\frac{(x'^2 + x' + z_b) + \sqrt{\sigma(x')}}{2x'}\right\}\frac{dx'}{x'} \\
&\simeq -\frac{w^2-1}{(1-Q_f)w}(\beta\Lambda)^2 + \frac{(1+Q_f)(w^4-1)}{4(1-Q_f)^3w^2}(\beta\Lambda)^4 - \frac{(1+4Q_f+Q_f^2)(w^6-1)}{9(1-Q_f)^5w^3}(\beta\Lambda)^6 \\
&\quad -\frac{2(w^2-1)}{(1-Q_f)^5w}(\beta\Lambda)^6 + \frac{(1+9Q_f+9Q_f^2+Q_f^3)(w^8-1)}{16(1-Q_f)^7w^4}(\beta\Lambda)^8 \\
&\quad +\frac{2(1+Q_f)(w^4-1)}{(1-Q_f)^7w^2}(\beta\Lambda)^8 + \mathcal{O}(\Lambda^{10}),
\end{aligned} \tag{7.7}$$

where in the second equality, since the toric brane is inserted on the base $\mathbf{P}^1_b$, we expanded the integrand around the midpoint $x' = z_b^{1/2}$ and took away a logarithmic term from the final result. When we compute the annulus and the three-holed amplitudes $\mathcal{F}^{(0,2)}, \mathcal{F}^{(0,3)}$ in the following, we will use a similar prescription as above. Using the relation (7.3) of geometric engineering and taking the limit $\beta \to 0$, we find a matching of (7.7) up to Λ^8 with the leading term (4.17) of the free energy obtained from the CFT one point function of $\Phi_{1,2}$.

We can compute the annulus amplitude $\mathcal{F}^{(0,2)}(x,y)$ using (6.8), where $G(k)$ can be rewritten in terms of the period $T_b = -\log Q_b$ as was shown in [58],

$$G(k) = -\frac{1}{12}\Delta_0(z_b, z_f)\widetilde{z}_b\frac{\partial}{\partial\widetilde{z}_b}\Big\{12\log\widetilde{z}_b\frac{\partial}{\partial\widetilde{z}_b}T_b + 4\log\widetilde{z}_b + \log\Delta_0(\widetilde{z}_b, \widetilde{z}_b\widetilde{z}_f)\Big\}, \tag{7.8}$$

where $\widetilde{z}_b = z_b, \widetilde{z}_f = z_f/z_b$, and $\Delta_0(z_b, z_f) := 1 - 8(z_b+z_f) + 16(z_b-z_f)^2$ is a component of the discriminant of the mirror curve (7.5). Thus we obtain the annulus amplitude

$$\begin{aligned}
&\mathcal{F}^{(0,2)}(Q_f, \Lambda, w_i) = \int^x\int^y B(x', y') - \frac{dx'dy'}{(x'-y')^2} \\
&\simeq \frac{Q_f(w_1^2w_2^2+1)}{(1-Q_f)^4w_1w_2}(\beta\Lambda)^4 - \frac{Q_f(1+Q_f)(w_1+w_2)(w_1^3w_2^3+1)}{(1-Q_f)^6w_1^2w_2^2}(\beta\Lambda)^6 \\
&+\frac{Q_f(1+3Q_f+Q_f^2)(w_1^2+w_2^2)(w_1^4w_2^4+1)}{(1-Q_f)^8w_1^3w_2^3}(\beta\Lambda)^8 + \frac{Q_f(2+5Q_f+2Q_f^2)(w_1^4w_2^4+1)}{2(1-Q_f)^8w_1^2w_2^2}(\beta\Lambda)^8 \\
&+\frac{2Q_f(1+6Q_f+Q_f^2)(w_1^2w_2^2+1)}{(1-Q_f)^8w_1w_2}(\beta\Lambda)^8 - \frac{2Q_f^2(w_1^2+w_2^2)}{(1-Q_f)^8w_1w_2}(\beta\Lambda)^8 + \mathcal{O}(\Lambda^{10}),
\end{aligned} \tag{7.9}$$

where in the second equality, we used a similar prescription to the case of the disk amplitude. The annulus amplitude with two arguments gives the contribution from a

geometry where two toric branes are inserted on the base $\mathbf{P}^1_b$. $X := \beta^2\Lambda^2 w_1^{-1}$ and $Y := \beta^2\Lambda^2 w_2^{-1}$ represent the positions of the first and the second toric brane, respectively. Using the relation (7.3) and taking the limit $\beta \to 0$, we find that (7.9) coincides with (6.17) and (B.7).

Higher topology amplitudes can be also computed by the topological recursion (6.7). As an example let us compute the three-holed amplitude $\mathcal{F}^{(0,3)}(x,y,z)$ using (6.19), where the moment function $M(x)$ is defined by

$$\begin{aligned} &\frac{1}{2}\big(\omega(x)-\omega(\bar{x})\big) = \frac{dx}{x}\tanh^{-1}\left[\frac{\sqrt{\sigma(x)}}{x^2+x+z_b}\right] =: M(x)\sqrt{\sigma(x)}dx, \\ \longrightarrow\quad &M(x) = \frac{1}{x\sqrt{\sigma(x)}}\tanh^{-1}\left[\frac{\sqrt{\sigma(x)}}{x^2+x+z_b}\right] = \frac{1}{x}\sum_{n=0}^{\infty}\frac{1}{2n+1}\frac{\sigma(x)^n}{(x^2+x+z_b)^{2n+1}}. \end{aligned} \tag{7.10}$$

Thus we obtain the three-holed amplitude

$$\begin{aligned} &\mathcal{F}^{(0,3)}(Q_f,\Lambda,w_i) = \int^x\int^y\int^z W^{(0,3)}(x',y',z') \\ &\simeq \frac{(Q_f+6Q_f^2+Q_f^3)(w_1^2w_2^2w_3^2-1)}{(1-Q_f)^7 w_1w_2w_3}(\beta\Lambda)^6 \\ &+\frac{(Q_f+11Q_f^2+11Q_f^3+Q_f^4)\big(w_1^3w_2^3w_3^3(w_1+w_2+w_3)-(w_1w_2+w_2w_3+w_3w_1)\big)}{(1-Q_f)^9 w_1^2w_2^2w_3^2}(\beta\Lambda)^8 \\ &+\Bigg\{\frac{2(Q_f^2+6Q_f^3+Q_f^4)\big(w_1^2+w_2^2+w_3^2-(w_1^2w_2^2+w_2^2w_3^2+w_3^2w_1^2)\big)}{(1-Q_f)^{11}w_1w_2w_3} \\ &+\frac{(Q_f+17Q_f^2+36Q_f^3+17Q_f^4+Q_f^5)\big(w_1^3w_2^3w_3^3(w_1w_2+w_2w_3+w_3w_1)-(w_1+w_2+w_3)\big)}{(1-Q_f)^{11}w_1^2w_2^2w_3^2} \\ &+\frac{(Q_f+18Q_f^2+42Q_f^3+18Q_f^4+Q_f^5)\big(w_1^4w_2^4w_3^4(w_1^2+w_2^2+w_3^2)-(w_1^2w_2^2+w_2^2w_3^2+w_3^2w_1^2)\big)}{(1-Q_f)^{11}w_1^3w_2^3w_3^3} \\ &+\frac{2(Q_f+28Q_f^2+86Q_f^3+28Q_f^4+Q_f^5)(w_1^2w_2^2w_3^2-1)}{(1-Q_f)^{11}w_1w_2w_3}\Bigg\}(\beta\Lambda)^{10}+\mathcal{O}(\Lambda^{12}). \end{aligned} \tag{7.11}$$

The three-holed amplitude with three arguments gives the leading contribution when three toric branes are inserted on the base $\mathbf{P}^1_b$. $X := \beta^2\Lambda^2 w_1^{-1}$, $Y := \beta^2\Lambda^2 w_2^{-1}$ and $Z := \beta^2\Lambda^2 w_3^{-1}$ represent the position of each toric brane. Using (7.3) and taking the limit $\beta \to 0$, we find that (7.11) agrees with (6.21) and (B.11).

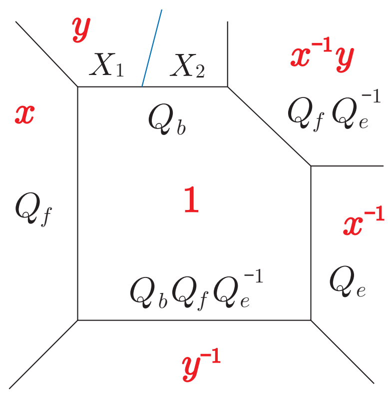

Figure 4: Local del Pezzo surface K_{dP_2}

7.2 Toric brane on local del Pezzo surface K_{dP_2}: $SU(2)$ with one fundamental matter

By the geometric engineering, one can also introduce fundamental matters. The $SU(2)$ theory with one fundamental matter is realized by the del Pezzo surface dP_2, which is obtained by a blow up at a torus fixed point on the Hirzebruch surface $\mathbf{F}_0$, . Let us insert a toric brane on the base $\mathbf{P}^1_b$ as the blue line in Fig.4. As in (7.3), the Kähler parameters Q_f, (resp. Q_b, Q_e) of the fiber $\mathbf{P}^1_f$, (resp. base $\mathbf{P}^1_b$, the exceptional curve $\mathbf{P}^1_e$), and the distance between the toric brane and the vertices in the web diagram $X_1, (X_2)$ can be related to the parameters on the gauge theory side as [18, 21],

$$Q_f = e^{-2\beta a}, \quad Q_e = e^{-\beta(a-m)}, \quad Q_b = X_1 X_2 = 2\beta^3\Lambda^3, \quad X_1 := \beta^2\Lambda^2 w^{-1}, \quad X_2 := 2\beta\Lambda w. \tag{7.12}$$

The charge vectors of K_{dP_2} are given by

$$\ell_b = (-1,1,0,0,1,-1), \quad \ell_f = (-2,0,1,0,0,1), \quad \ell_e = (-1,0,1,-1,1,0). \tag{7.13}$$

By taking the local coordinate patch as Fig.4, we obtain the mirror curve which describes the moduli of the toric brane,

$$\begin{aligned} &(x+z_b)y^2 + (x^2 + x + z_b z_f z_e^{-1})y + z_f x = 0, \\ &\sigma(x) := (x^2 + x + z_b z_f z_e^{-1})^2 - 4 z_f x(x+z_b), \end{aligned} \tag{7.14}$$

where z_f, z_b, and z_e are the moduli parameters of complex structure the mirror Calabi-Yau threefold. The closed and open mirror maps are given by [57],

$$
\begin{aligned}
&\log Q_b = \log z_b + \sum_{m,n,k\geq 0,(m,n,k)\neq(0,0,0)}^{\infty} (-1)^m \frac{(3m+2n+2k-1)!}{m!n!k!(m+k)!(m+n)!} z_b^{m+k} z_f^{m+n+k} z_e^{-k}, \\
&\log \frac{Q_f}{Q_b^2} = \log \frac{z_f}{z_b^2}, \quad \log \frac{Q_e}{Q_b} = \log \frac{z_e}{z_b}, \quad X = \frac{Q_b}{z_b} x,
\end{aligned}
\tag{7.15}
$$

where $X = X(x)$ is the open string moduli on the A model side. The disk amplitude is computed as

$$
\begin{aligned}
&\mathcal{F}^{(0,1)}(Q_f, Q_e, \Lambda, w) = \int^x \omega(x') = \int^x \log\left\{\frac{(x'^2 + x' + z_b z_f z_e^{-1}) + \sqrt{\sigma(x')}}{2(x'+z_b)}\right\}\frac{dx'}{x'} \\
&\simeq -\frac{2(Q_f - Q_e)w}{(1-Q_f)Q_e}\beta\Lambda + \frac{1}{(1-Q_f)w}(\beta\Lambda)^2 + \frac{(Q_f-Q_e)(Q_f^2+Q_f+Q_e-3Q_fQ_e)w^2}{(1-Q_f)^3Q_e^2}(\beta\Lambda)^2 \\
&-\frac{8(Q_f-Q_e)\big(Q_f^4+4Q_f^3+Q_f^2-(8Q_f^3+5Q_f^2-Q_f)Q_e+(10Q_f^2-5Q_f+1)Q_e^2\big)w^3}{9(1-Q_f)^5Q_e^3}(\beta\Lambda)^3 \\
&+\Bigg\{\frac{(Q_f-Q_e)}{(1-Q_f)^7Q_e^4}\Big((Q_f+1)(Q_f^2+8Q_f+1)Q_f^3-(15Q_f^3+39Q_f^2+7Q_f-1)Q_f^2Q_e \\
&\qquad +(45Q_f^3+21Q_f^2-7Q_f+1)Q_fQ_e^2-(35Q_f^3-21Q_f^2+7Q_f-1)Q_e^3\Big)w^4 \\
&\quad -\frac{8Q_f^2(Q_f-Q_e)(1-Q_e)w}{(1-Q_f)^5Q_e^2} - \frac{1+Q_f}{4(1-Q_f)^3w^2}\Bigg\}(\beta\Lambda)^4 + \mathcal{O}(\Lambda^5),
\end{aligned}
\tag{7.16}
$$

where in the second equality we expanded the integrand around the midpoint $x' = z_b^{1/2}$ and removed a logarithmic term. We see that this result is consistent with (7.7) in the limit $Q_e \to 0, Q_f/Q_e \to 1$. Using the relation (7.12) and taking the limit $\beta \to 0$, we find that (7.16) agrees with (4.31) except the coefficient of $w\Lambda$. The difference of the coefficient of $w\Lambda$ is nothing but the overall factor $\exp(-\Lambda w/\hbar)$ which compensates the difference between the instanton partition function with surface operator and the correlation function with the degenerate primary field insertion. Therefore (7.16) agrees with the computation on the gauge theory side.

From (6.8) we can also compute the annulus amplitude $\mathcal{F}^{(0,2)}(x,y)$, where $G(k)$ can

be rewritten in terms of the period $T_b = -\log Q_b$ as [58],

$$
\begin{aligned}
G(k) &= -\frac{\Delta_0(z_b,z_f,z_e)}{24z_e^2 C(z_b,z_f,z_e)}\widetilde{z}_b\frac{\partial}{\partial\widetilde{z}_b}\Big\{12\log\widetilde{z}_b\frac{\partial}{\partial\widetilde{z}_b}T_b + 7\log\widetilde{z}_b + \log\Delta_0(\widetilde{z}_b,\widetilde{z}_b^2\widetilde{z}_f,\widetilde{z}_b\widetilde{z}_e)\Big\},\\
\Delta_0(z_b,z_f,z_e) &:= 16z_b^3z_f^2z_e - \big(27z_e^4 - 12(3z_e-2z_f)z_e^2 + 8(z_e^2+2z_fz_e-2z_f^2)\big)z_b^2z_f\\
&\quad+\big(12(3z_e-2z_f)z_fz_e^2 - (z_e^3+46z_fz_e^2-64z_f^2z_e+32z_f^3) + (z_e^2+8z_fz_e-8z_f^2)\big)z_bz_e\\
&\quad+(4z_f-1)^2(z_e^2-z_e+z_f)z_e^2,\\
C(z_b,z_f,z_e) &:= -\frac{9}{8}z_bz_e^2 + (z_b+z_e)z_e - \frac{1}{8}\big(7z_e+4z_f(z_b+z_e)\big) + z_f, \qquad (7.17)
\end{aligned}
$$

where $\widetilde{z}_b = z_b, \widetilde{z}_f = z_f/z_b^2, \widetilde{z}_e = z_e/z_b$, and $\Delta_0(z_b,z_f,z_e)$ is a component of the discriminant of the mirror curve (7.14). Thus we obtain the annulus amplitude

$$
\begin{aligned}
&\mathcal{F}^{(0,2)}(Q_f,Q_e,\Lambda,w_i) = \int^x\int^y B(x',y') - \frac{dx'dy'}{(x'-y')^2}\\
&\simeq \frac{4Q_f^2(Q_f-Q_e)(1-Q_e)w_1w_2}{(1-Q_f)^4Q_e^2}(\beta\Lambda)^2\\
&-\frac{8Q_f^3(Q_f-Q_e)(1-Q_e)(Q_f+1-2Q_e)w_1w_2(w_1+w_2)}{(1-Q_f)^6Q_e^3}(\beta\Lambda)^3\\
&+\Bigg\{\frac{16Q_f^4(Q_f-Q_e)(1-Q_e)\big(Q_f^2+3Q_f+1+5(Q_e-Q_f-1)Q_e\big)(w_1^2+w_2^2)w_1w_2}{(1-Q_f)^8Q_e^4}\\
&\quad+\frac{8Q_f^4(Q_f-Q_e)(1-Q_e)(2Q_f+1-3Q_e)(Q_f+2-3Q_e)w_1^2w_2^2}{(1-Q_f)^8Q_e^4}\\
&\quad+\frac{Q_f}{(1-Q_f)^4w_1w_2}\Bigg\}(\beta\Lambda)^4 + \mathcal{O}(\Lambda^5), \qquad (7.18)
\end{aligned}
$$

where in the second equality we expanded the Bergman kernel around the point $x' = z_b^{1/2}, y' = z_b^{1/2}$ and removed a logarithmic term. $X := \beta^2\Lambda^2w_1^{-1}$, and $Y := \beta^2\Lambda^2w_2^{-1}$ represent the position of two toric branes. Using the geometric engineering (7.12) and taking the limit $\beta\to 0$, we see that (7.18) agrees with (6.24), and (B.16).

8 Vortex counting and open topological A model

In a recent paper [8] the instanton partition function with surface operator has been worked out from the viewpoint of the coupling of four dimensional gauge theory with a two dimensional theory on the surface. It was argued that in the decoupling limit $\Lambda_{\text{inst}} \to 0$, where only zero instanton sector of the four dimensional theory survives, the

partition function reduces to the vortex counting in the two dimensional theory. By the localization computation on the affine Laumon space which we used in section 3, the vortex counting of [8] can be derived in the following way. Recall our identification of the instanton number k and the monopole number $\mathfrak{m}$:

$$k = k_1, \qquad \mathfrak{m} = k_2 - k_1, \tag{8.1}$$

where k_1 and k_2 are given by (3.5) in terms of a pair of Young diagrams (λ_1, λ_2). Thus when we look at the zero instanton sector we have to set $k_1 = 0$ and the monopole number is restricted to take non-negative integer[9]. This means that λ_1 has to be trivial and λ_2 has only a single row, whose length gives the monopole number. Let us first consider pure $SU(2)$ theory for simplicity. Then almost all terms in the diagonal part of the equivariant character vanish. The remaining terms are

$$\begin{aligned}\mathrm{ch}_{\vec{\lambda},\vec{\lambda}}(a,a) &= e^{a_1-a_2+\epsilon_1+\epsilon_2}\frac{e^{\epsilon_1\lambda_{2,1}}-1}{e^{\epsilon_1}-1} - e^{\epsilon_1}\frac{(e^{\epsilon_1\lambda_{2,1}}-1)(e^{-\epsilon_1\lambda_{2,1}}-1)}{e^{\epsilon_1}-1} - e^{\epsilon_1}\frac{(e^{-\epsilon_1\lambda_{2,1}}-1)}{e^{\epsilon_1}-1} \\ &= \left(e^{a_1-a_2+\epsilon_1+\epsilon_2}+e^{\epsilon_1}\right)\frac{e^{\epsilon_1\mathfrak{m}}-1}{e^{\epsilon_1}-1} = \left(e^{a_1-a_2+\epsilon_2}+1\right)\sum_{k=1}^{\mathfrak{m}} e^{k\epsilon_1}.\end{aligned} \tag{8.2}$$

Hence the zero instanton part of the partition function (3.8) is

$$Z_{\mathrm{monopole}}(a,\epsilon_1,\epsilon_2;z) = 1 + \sum_{\mathfrak{m}=1}^{\infty}\frac{1}{\mathfrak{m}!}\left(\frac{z}{\epsilon_1}\right)^{\mathfrak{m}}\prod_{k=1}^{\mathfrak{m}}\frac{1}{2a+\epsilon_2+k\epsilon_1}, \tag{8.3}$$

where we have replaced the parameter Λ_2 in (3.8) to z. We see that with the choice of equivariant parameters $\epsilon_1 = \hbar, \epsilon_2 \to 0$ the partition function (8.3) agrees with the generating function of vortex counting, eq. (3.24) in [8], where it was argued that the K theory version of (3.24) coincides with a refined open topological string amplitude in the limit where the Kähler parameter Q_b of the base $\mathbf{P}^1_b$ vanishes.

Let us look at a similar vortex counting in zero instanton background in $N_f = 4$ theory. The results for $1 \le N_f \le 3$ theories can be obtained by the decoupling limit. The contributions from $(\emptyset, (\mathfrak{m}))$ gives the partition function

$$Z^{N_f=4}_{\mathrm{monopole}}(a, M_i, \epsilon_1, \epsilon_2; y) = 1 + \sum_{\mathfrak{m}=1}^{\infty} y^{\mathfrak{m}}\frac{n_f^S[\vec{\emptyset},(\emptyset,(\mathfrak{m}))](a_1, a\ ; m_1)\cdot n_f^S[(\emptyset,(\mathfrak{m})),\vec{\emptyset}](a,a_2\ ; m_2)}{n_{\mathrm{vec}}[(\emptyset,(\mathfrak{m}))](\vec{a})}. \tag{8.4}$$

[9] As we will see the next subsection, for $k > 0$ the negative monopole number is allowed.

Since only the non-vanishing component of $\vec{\lambda}$ is $\lambda_{2,1} = \mathfrak{m}$, we have

$$n_f^S[(\emptyset, (\mathfrak{m})), \vec{\emptyset}](a, b\ ; m) = \prod_{k=0}^{\mathfrak{m}-1}(-a + b - k\epsilon_1 - m), \tag{8.5}$$

$$n_f^S[\vec{\emptyset}, (\emptyset, (\mathfrak{m}))](a, b\ ; m) = \prod_{k=1}^{\mathfrak{m}}(a + b + k\epsilon_1 + \epsilon_2 - m). \tag{8.6}$$

Combined with the previous computation of $n_{\mathrm{vec}}[(\emptyset, (\mathfrak{m}))](\vec{a})$ in pure $SU(2)$ theory, this gives

$$Z^{N_f=4}_{\mathrm{monopole}}(a, M_i, \epsilon_1, \epsilon_2; y) = 1 + \sum_{\mathfrak{m}=1}^{\infty}(-y)^{\mathfrak{m}} \frac{\prod_{k=1}^{\mathfrak{m}}(a - 2M_2 + k\epsilon_1 + \epsilon_2)(a + 2M_4 + (k-1)\epsilon_1)}{\mathfrak{m}!\epsilon_1^{\mathfrak{m}} \prod_{k=1}^{\mathfrak{m}}(2a + k\epsilon_1 + \epsilon_2)} \tag{8.7}$$

After the identification $z \equiv (-y), 2M_2 \equiv m_1 - \frac{1}{2}\epsilon_1 + \frac{1}{2}\epsilon_2, 2M_4 \equiv -m_2 + \frac{3}{2}\epsilon_1 + \frac{1}{2}\epsilon_2$, (8.7) agrees to (4.26) in [8] up to $U(1)$ factor (an appropriate power of $(1-z)$), which is also related to the hypergeometric series.

8.1 Localization on one instanton sector

We generalize the vortex counting to one instanton sector of the four dimensional gauge theory. For one instanton sector, the pair of partitions $\vec{\lambda} := (\lambda_1, \lambda_2)$ satisfies

$$k_1 = 1, \quad k_2 = \mathfrak{m} + 1. \tag{8.8}$$

There are four choices for $\vec{\lambda}$ as follows:

$$\begin{aligned} &(A) \quad \vec{\lambda}_{\mathfrak{m}A} = ((1), (\mathfrak{m}+1)), \quad \mathfrak{m} \geq -1, \qquad (B) \quad \vec{\lambda}_{\mathfrak{m}B} = (\emptyset, (\mathfrak{m}+1, 1)), \quad \mathfrak{m} \geq 0, \\ &(C) \quad \vec{\lambda}_{\mathfrak{m}C} = (\emptyset, (\mathfrak{m}, 1, 1)), \quad \mathfrak{m} \geq 1, \qquad (D) \quad \vec{\lambda}_{\mathfrak{m}D} = ((1,1), (\mathfrak{m})), \quad \mathfrak{m} \geq 0. \end{aligned} \tag{8.9}$$

For $N_f = 4$ theory, by evaluating the three characters $\mathrm{Tr}_{\mathrm{Ext}(\vec{\lambda},\vec{\lambda})}[g]$, $\mathrm{Tr}_{\mathrm{Ext}(\vec{\lambda},\vec{\emptyset})}[g]$ with $a_1 = a$, $a_2 = -a$, $b_1 = M_3 - M_4$, $b_2 = M_4 - M_3$, $m_2 = M_3 + M_4$ and $\mathrm{Tr}_{\mathrm{Ext}(\vec{\emptyset},\vec{\lambda})}[g]$ with $a_1 = M_1 - M_2$, $a_2 = M_2 - M_1$, $b_1 = a$, $b_2 = -a$, $m_1 = M_1 + M_2$ we can read off the

partition function in one instanton sector for each partition.

$$Z_1^{(A)}(a, M_i, \epsilon_1, \epsilon_2; x, y) = \sum_{\mathfrak{m}=-1}^{\infty} xy^{\mathfrak{m}+1} \frac{n_f^S[\vec{\emptyset}, \vec{\lambda}_{\mathfrak{m}A}](a_1, a; m_1) \cdot n_f^S[\vec{\lambda}_{\mathfrak{m}A}, \vec{\emptyset}](a_1, a; m_2)}{n_{\text{vec}}[\vec{\lambda}_{\mathfrak{m}A}](\vec{a})}, \tag{8.10}$$

$$n_f^S(\vec{\lambda}_{\mathfrak{m}A}, \vec{\emptyset}) = (a - 2M_3) \prod_{k=0}^{\mathfrak{m}} (-a - 2M_4 - k\epsilon_1),$$

$$n_f^S(\vec{\emptyset}, \vec{\lambda}_{\mathfrak{m}A}) = (-a - 2M_1 + \epsilon_1) \prod_{k=1}^{\mathfrak{m}+1} (a - 2M_2 + k\epsilon_1 + \epsilon_2),$$

$$n_{\text{vec}}[\vec{\lambda}_{\mathfrak{m}A}](\vec{a}) = (-2a - \mathfrak{m}\epsilon_1)\epsilon_1^{\mathfrak{m}+2}(\mathfrak{m}+1)! \prod_{k=0}^{\mathfrak{m}} (2a + k\epsilon_1 + \epsilon_2),$$

$$Z_1^{(B)}(a, M_i, \epsilon_1, \epsilon_2; x, y) = \sum_{\mathfrak{m}=0}^{\infty} xy^{\mathfrak{m}+1} \frac{n_f^S[\vec{\emptyset}, \vec{\lambda}_{\mathfrak{m}B}](a_1, a; m_1) \cdot n_f^S[\vec{\lambda}_{\mathfrak{m}B}, \vec{\emptyset}](a_1, a; m_2)}{n_{\text{vec}}[\vec{\lambda}_{\mathfrak{m}B}](\vec{a})}, \tag{8.11}$$

$$n_f^S(\vec{\lambda}_{\mathfrak{m}B}, \vec{\emptyset}) = (-a - 2M_3 - \epsilon_2) \prod_{k=0}^{\mathfrak{m}} (-a - 2M_4 - k\epsilon_1),$$

$$n_f^S(\vec{\emptyset}, \vec{\lambda}_{\mathfrak{m}B}) = (a - 2M_1 + \epsilon_1 + \epsilon_2) \prod_{k=1}^{\mathfrak{m}+1} (a - 2M_2 + k\epsilon_1 + \epsilon_2),$$

$$n_{\text{vec}}[\vec{\lambda}_{\mathfrak{m}B}](\vec{a}) = (\mathfrak{m}\epsilon_1 - \epsilon_2)\epsilon_1^{\mathfrak{m}+1}\mathfrak{m}! \prod_{k=0}^{\mathfrak{m}+1} (2a + k\epsilon_1 + \epsilon_2),$$

$$Z_1^{(C)}(a, M_i, \epsilon_1, \epsilon_2; x, y) = \sum_{\mathfrak{m}=1}^{\infty} xy^{\mathfrak{m}+1} \frac{n_f^S[\vec{\emptyset}, \vec{\lambda}_{\mathfrak{m}C}](a_1, a; m_1) \cdot n_f^S[\vec{\lambda}_{\mathfrak{m}C}, \vec{\emptyset}](a_1, a; m_2)}{n_{\text{vec}}[\vec{\lambda}_{\mathfrak{m}C}](\vec{a})}, \tag{8.12}$$

$$n_f^S(\vec{\lambda}_{\mathfrak{m}C}, \vec{\emptyset}) = (-a - 2M_3 - \epsilon_2)(-a - 2M_4 - \epsilon_2) \prod_{k=0}^{\mathfrak{m}-1} (-a - 2M_4 - k\epsilon_1)$$

$$n_f^S(\vec{\emptyset}, \vec{\lambda}_{\mathfrak{m}C}) = (a - 2M_1 + \epsilon_1 + \epsilon_2)(a - 2M_2 + \epsilon_1 + 2\epsilon_2) \prod_{k=1}^{\mathfrak{m}} (a - 2M_2 + k\epsilon_1 + \epsilon_2),$$

$$n_{\text{vec}}[\vec{\lambda}_{\mathfrak{m}C}](\vec{a}) = (2a + \epsilon_1 + 2\epsilon_2)(-\mathfrak{m}\epsilon_1 + \epsilon_2)\epsilon_2\epsilon_1^{\mathfrak{m}}(\mathfrak{m}-1)! \prod_{k=0}^{\mathfrak{m}} (2a + k\epsilon_1 + \epsilon_2),$$

$$Z_1^{(D)}(a,M_i,\epsilon_1,\epsilon_2;x,y)=\sum_{\mathfrak{m}=0}^{\infty}xy^{\mathfrak{m}+1}\frac{n_f^S[\vec{\emptyset},\vec{\lambda}_{\mathfrak{m}D}](a_1,a;m_1)\cdot n_f^S[\vec{\lambda}_{\mathfrak{m}D},\vec{\emptyset}](a_1,a;m_2)}{n_{\mathrm{vec}}[\vec{\lambda}_{\mathfrak{m}D}](\vec{a})}, \quad (8.13)$$

$$n_f^S(\vec{\lambda}_{\mathfrak{m}D},\vec{\emptyset})=(a-2M_3)(a-2M_4)\prod_{k=0}^{\mathfrak{m}-1}(-a-2M_4-k\epsilon_1),$$

$$n_f^S(\vec{\emptyset},\vec{\lambda}_{\mathfrak{m}D})=(a+2M_1-\epsilon_1)(a+2M_2-\epsilon_1-\epsilon_2)\prod_{k=1}^{\mathfrak{m}}(a-2M_2+k\epsilon_1+\epsilon_2),$$

$$n_{\mathrm{vec}}[\vec{\lambda}_{\mathfrak{m}D}](\vec{a})=(-2a+\epsilon_1)(2a+\mathfrak{m}\epsilon_1)\epsilon_2\epsilon_1^{\mathfrak{m}+1}\mathfrak{m}!\prod_{k=0}^{\mathfrak{m}-1}(2a+k\epsilon_1+\epsilon_2).$$

Taking a decoupling limit

$$M_2,M_3,M_4\to\infty,\quad \Lambda_1=-2M_3x,\quad \Lambda_2=4M_2M_4y, \quad (8.14)$$

one obtains the partition function for $N_f=1$ theory

$$\begin{aligned}
&Z_1^{(A)}(a,M_1,\epsilon_1,\epsilon_2;\Lambda_1,\Lambda_2)\\
&\quad=\sum_{\mathfrak{m}=-1}^{\infty}\Lambda_1\Lambda_2^{\mathfrak{m}+1}\frac{a+2M_1-\epsilon_1}{(2a+\mathfrak{m}\epsilon_1)\epsilon_1^{\mathfrak{m}+2}(\mathfrak{m}+1)!\prod_{k=0}^{\mathfrak{m}}(2a+k\epsilon_1+\epsilon_2)},\\
&Z_1^{(B)}(a,M_1,\epsilon_1,\epsilon_2;\Lambda_1,\Lambda_2)\\
&\quad=\sum_{\mathfrak{m}=0}^{\infty}\Lambda_1\Lambda_2^{\mathfrak{m}+1}\frac{a-2M_1+\epsilon_1+\epsilon_2}{(\mathfrak{m}\epsilon_1-\epsilon_2)\epsilon_1^{\mathfrak{m}+1}\mathfrak{m}!\prod_{k=0}^{\mathfrak{m}+1}(2a+k\epsilon_1+\epsilon_2)},\\
&Z_1^{(C)}(a,M_1,\epsilon_1,\epsilon_2;\Lambda_1,\Lambda_2)\\
&\quad=\sum_{\mathfrak{m}=1}^{\infty}\Lambda_1\Lambda_2^{\mathfrak{m}+1}\frac{a-2M_1+\epsilon_1+\epsilon_2}{(2a+\epsilon_1+2\epsilon_2)(-\mathfrak{m}\epsilon_1+\epsilon_2)\epsilon_2\epsilon_1^{\mathfrak{m}}(\mathfrak{m}-1)!\prod_{k=0}^{\mathfrak{m}}(2a+k\epsilon_1+\epsilon_2)},\\
&Z_1^{(D)}(a,M_1,\epsilon_1,\epsilon_2;\Lambda_1,\Lambda_2)\\
&\quad=\sum_{\mathfrak{m}=0}^{\infty}\Lambda_1\Lambda_2^{\mathfrak{m}+1}\frac{a+2M_1-\epsilon_1}{(2a-\epsilon_1)(2a+\mathfrak{m}\epsilon_1)\epsilon_2\epsilon_1^{\mathfrak{m}+1}\mathfrak{m}!\prod_{k=0}^{\mathfrak{m}-1}(2a+k\epsilon_1+\epsilon_2)}.
\end{aligned} \quad (8.15)$$

The decoupling limit

$$M_1,M_2,M_3,M_4\to\infty,\quad \Lambda_1=4M_1M_3x,\quad \Lambda_2=4M_2M_4y, \quad (8.16)$$

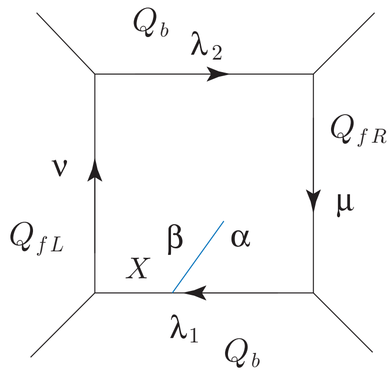

Figure 5: D-brane inserted on an inner leg of local Hirzebruch surface

gives the one instanton partition function for $N_f = 0$ theory

$$
\begin{aligned}
&Z_1^{(A)}(a, \epsilon_1, \epsilon_2; \Lambda_1, \Lambda_2) \\
&\quad = -\sum_{\mathfrak{m}=-1}^{\infty} \Lambda_1 \Lambda_2^{\mathfrak{m}+1} \frac{1}{(2a + \mathfrak{m}\epsilon_1)\epsilon_1^{\mathfrak{m}+2}(\mathfrak{m}+1)! \prod_{k=0}^{\mathfrak{m}}(2a + k\epsilon_1 + \epsilon_2)}, \\
&Z_1^{(B)}(a, \epsilon_1, \epsilon_2; \Lambda_1, \Lambda_2) \\
&\quad = \sum_{\mathfrak{m}=0}^{\infty} \Lambda_1 \Lambda_2^{\mathfrak{m}+1} \frac{1}{(\mathfrak{m}\epsilon_1 - \epsilon_2)\epsilon_1^{\mathfrak{m}+1}\mathfrak{m}! \prod_{k=0}^{\mathfrak{m}+1}(2a + k\epsilon_1 + \epsilon_2)}, \\
&Z_1^{(C)}(a, \epsilon_1, \epsilon_2; \Lambda_1, \Lambda_2) \\
&\quad = \sum_{\mathfrak{m}=1}^{\infty} \Lambda_1 \Lambda_2^{\mathfrak{m}+1} \frac{1}{(-\mathfrak{m}\epsilon_1 + \epsilon_2)(2a + \epsilon_1 + 2\epsilon_2)\epsilon_2\epsilon_1^{\mathfrak{m}}(\mathfrak{m}-1)! \prod_{k=0}^{\mathfrak{m}}(2a + k\epsilon_1 + \epsilon_2)}, \\
&Z_1^{(D)}(a, \epsilon_1, \epsilon_2; \Lambda_1, \Lambda_2) \\
&\quad = \sum_{\mathfrak{m}=0}^{\infty} \Lambda_1 \Lambda_2^{\mathfrak{m}+1} \frac{1}{(2a - \epsilon_1)(2a + \mathfrak{m}\epsilon_1)(-\epsilon_2)\epsilon_1^{\mathfrak{m}+1}\mathfrak{m}! \prod_{k=0}^{\mathfrak{m}-1}(2a + k\epsilon_1 + \epsilon_2)}.
\end{aligned}
\tag{8.17}
$$

8.2 Topological vertex computation

Now we discuss the one instanton partition function for the four dimensional gauge theory from the A model via geometric engineering. The pure gauge theory is engineered by the A model on the local Hirzebruch surface $\mathbf{F}_0 = \mathbf{P}_b^1 \times \mathbf{P}_f^1$ with a toric brane. The Kähler parameters for the base $\mathbf{P}_b^1$ and the fiber $\mathbf{P}_f^1$ correspond to $Q_b = \beta^4\Lambda^4$ and $Q_f = e^{-2\beta a}$, respectively. To realize the surface operator in four dimensional theory, the toric brane should be inserted on the inner leg which denotes the base $\mathbf{P}_b^1$ in the toric diagram

[25, 6, 8], and we choose the open string moduli $X = \beta^2\Lambda^2 w$.

The topological vertex computes the open BPS invariants [30, 59].

$$Z_{\rm BPS}^{\rm open}(X, Q_b, Q_{fL}, Q_{fR}; q) = \frac{Z^{\text{D-brane}}(X, Q_b, Q_{fL}, Q_{fR}; q)}{Z_{\rm BPS}^{\rm closed}(Q_b, Q_{fL}, Q_{fR}; q)}. \tag{8.18}$$

Each factor is given by the representation sums

$$\begin{aligned}
&Z_{\rm BPS}^{\rm closed}(Q_b, Q_{fL}, Q_{fR}; q) \\
&= \sum_{\mu,\nu,\lambda_1,\lambda_2} C_{\lambda_1\mu^t\emptyset} C_{\nu\lambda_1^t\emptyset} C_{\lambda_2\nu^t\emptyset} C_{\mu\lambda_2^t\emptyset} q^{(\kappa_\nu+\kappa_{\lambda_2}+\kappa_\mu+\kappa_{\lambda_1})/2} Q_{fL}^{|\nu|} Q_{fR}^{|\mu|} Q_b^{|\lambda_1|+|\lambda_2|}, \qquad (8.19)\\
&Z^{\text{D-brane}}(X, Q_b, Q_{fL}, Q_{fR}; q) \\
&= \sum_{\mu,\nu,\lambda_1,\lambda_2,\alpha,\beta} C_{(\lambda_1\otimes\alpha)\mu^t\emptyset} C_{\nu(\lambda_1^t\otimes\beta)\emptyset} C_{\lambda_2\nu^t\emptyset} C_{\mu\lambda_2^t\emptyset} q^{(\kappa_\nu+\kappa_{\lambda_2}+\kappa_\mu+(p-1)\kappa_{\lambda_1^t\otimes\beta}+p\kappa_{\lambda_1\otimes\alpha})/2} \\
&\qquad \times Q_{fL}^{|\nu|} Q_{fR}^{|\mu|} Q_b^{|\lambda_1|+|\lambda_2|+|\beta|} (-1)^{|\lambda_1|+(p-1)|\lambda_1^t\otimes\beta|+p|\lambda_1\otimes\alpha|} \ {\rm Tr}_\alpha V \ {\rm Tr}_\beta V^{-1}, \qquad (8.20)
\end{aligned}$$

where $V = X$ and p denotes the framing of the toric brane. For the tensor product representation $\alpha\otimes\beta$, $\kappa_{\alpha\otimes\beta} = \kappa_\alpha + \kappa_\beta$ and $C_{(\alpha\otimes\beta)\mu\nu} = \sum_\gamma c_{\alpha\beta}^\gamma C_{\gamma\mu\nu}$ where $c_{\alpha\beta}^\gamma$ is Littlewood-Richardson coefficient [60]. In the following we choose the framing $p = -1$. In order to compare with the four dimensional gauge theory in detail, we have set the Kähler parameter for the fiber $\mathbf{P}_f^1$ in the left/right side in the toric diagram independently as $Q_{fL} = e^{-2\beta a_L}$ and $Q_{fR} = e^{-2\beta a_R}$.

The one instanton part of the topological string amplitude is the first order in Q_b^1. For the closed string partition function $Z_{\rm BPS}^{\rm closed}(Q_b, Q_{fL}, Q_{fR}; q)$, we only need to consider the terms with $\lambda_1 = \lambda_2 = \emptyset$. For such choice of the partitions, one finds the closed string partition function

$$Z_{\rm BPS\,0}^{\rm closed}(Q_b, Q_{fL}, Q_{fR}; q) = M(Q_{fL}; q) M(Q_{fR}; q), \tag{8.21}$$

$$M(Q; q) = \prod_{n=1}^{\infty} (1 - Qq^n)^{-n}. \tag{8.22}$$

On the other hand, the D-brane partition function $Z^{\text{D-brane}}(X, Q_b, Q_{fL}, Q_{fR}; q)$ in one instanton sector comes from the following three choices of the partitions:

$$(1)\ (\beta, \lambda_1, \lambda_2) = (\Box, \emptyset, \emptyset), \quad (2)\ (\beta, \lambda_1, \lambda_2) = (\emptyset, \emptyset, \Box), \quad (3)\ (\beta, \lambda_1, \lambda_2) = (\emptyset, \Box, \emptyset). \tag{8.23}$$

At this point we should point out a crucial difference from the case of the geometric engineering of the Nekrasov partition function in terms of closed topological string. In the case of the Nekrasov partition function for $SU(N)$ gauge theory, the fixed points on the instanton moduli space are in one to one correspondence with the assignments of the Young diagrams on N parallel inner edges representing the base $\mathbf{P}^1_b$ of ALE fibration of type A_{N-1}. However, in the present case even at one instanton level the one to one correspondence is lost. In fact we found four fixed points (8.9) on the affine Laumon space with instanton number one, while (8.23) gives only three configurations. The lack of one to one correspondence makes the problem of matching the instanton partition function with surface operator to open topological string amplitudes highly non-trivial.

Let us compute the open BPS partition function for each choice of partitions. For case (1), the open BPS partition function $Z^{\text{open (1)}}_{\text{BPS 1}}(X, Q_b, Q_{fL}, Q_{fR}; q)$ becomes

$$\begin{aligned}
&Z^{\text{open (1)}}_{\text{BPS 1}}(X, Q_b, Q_{fL}, Q_{fR}; q) \\
&= \frac{1}{Z^{\text{closed}}_{\text{BPS 0}}(Q_b, Q_{fL}, Q_{fR}; q)} \\
&\quad \times Q_b \sum_{\alpha} s_\alpha(q^{-\rho}) s_\alpha(X) s_\square(q^\rho) q^{-\kappa_\square/2} \sum_{\mu} s_\mu(q^{\rho+\alpha}) s_\mu(q^\rho) Q_{fR}^{|\mu|} \\
&\quad \times \sum_{\nu} s_\nu(q^{\rho+\square}) s_\nu(q^\rho) Q_{fL}^{|\nu|} s_\square(X^{-1}) \\
&= Q_b \frac{q^{1/2}}{q-1} \frac{1}{1-Q_{fL}} \sum_{\mathfrak{m}'=0}^{\infty} \frac{X^{-1}(Xq^{1/2})^{\mathfrak{m}'}}{\prod_{k=1}^{\mathfrak{m}'}(1-q^k)(1-Q_{fR}q^{k-1})}.
\end{aligned} \tag{8.24}$$

In the computation, we used the following relations.

$$s_{(\mathfrak{m})}(q^{-\rho}) = \frac{q^{\mathfrak{m}/2}}{\prod_{k=1}^{\mathfrak{m}}(1-q^k)}, \quad \mathrm{Tr}_{(\mathfrak{m})} X = s_{(\mathfrak{m})}(X) = X^{\mathfrak{m}}, \tag{8.25}$$

$$\sum_{\mu} s_\mu(q^{\rho+(\mathfrak{m})}) s_\mu(q^\rho) Q^{|\mu|} = M(Q;q) \prod_{k=1}^{\mathfrak{m}} \frac{1}{1-Qq^{k-1}}. \tag{8.26}$$

For case (2), we obtain

$$
\begin{aligned}
&Z^{\text{open }(2)}_{\text{BPS }1}(X, Q_b, Q_{fL}, Q_{fR}; q) \\
&= \frac{1}{Z^{\text{closed}}_{\text{BPS }0}(Q_b, Q_{fL}, Q_{fR}; q)} \\
&\qquad \times Q_b \sum_{\alpha} s_\alpha(q^{-\rho}) s_\alpha(X) s_\Box(q^\rho)^2 \sum_\mu s_\mu(q^{\rho+\alpha}) s_\mu(q^{\rho+\Box}) Q_{fR}^{|\mu|} \sum_\nu s_\nu(q^\rho) s_\nu(q^{\rho+\Box}) Q_{fL}^{|\nu|} \\
&= Q_b \frac{q}{(q-1)^2} \frac{1}{1-Q_{fL}} \sum_{\mathfrak{m}=0}^{\infty} \frac{1}{(1-Q_{fR}q^{-1})(1-Q_{fR}q^{\mathfrak{m}})} \frac{(Xq^{1/2})^{\mathfrak{m}}}{\prod_{k=1}^{\mathfrak{m}}(1-q^k)\prod_{k=1}^{\mathfrak{m}-1}(1-Q_{fR}q^{k-1})}.
\end{aligned}
\tag{8.27}
$$

To derive this result, we applied a relation

$$
\sum_\mu s_\mu(q^{\rho+(\mathfrak{m})}) s_\mu(q^{\rho+\Box}) Q^{|\mu|} = M(Q;q) \frac{1}{(1-Qq^{-1})(1-Qq^{\mathfrak{m}})} \frac{1}{\prod_{k=1}^{\mathfrak{m}-1}(1-Qq^{k-1})}. \tag{8.28}
$$

This is found from the Cauchy formula (C.5).

For case (3), we have to consider the topological vertex with a tensor product representation $C_{(\alpha\otimes\Box)\mu^t\emptyset}$ seriously. For the tensor product representation $\alpha \otimes \beta$, the Schur function obeys [60]

$$
s_{\alpha\otimes\beta} = s_\alpha s_\beta = \sum_\gamma c^\gamma_{\alpha\beta} s_\gamma. \tag{8.29}
$$

Then the topological vertex with a tensor product representation is computed as

$$
\begin{aligned}
C_{\emptyset(\alpha\otimes\beta)\mu} &= \sum_\gamma c^\gamma_{\alpha\beta} C_{\emptyset\gamma\mu} = \sum_\gamma c^\gamma_{\alpha\beta} s_\gamma(q^\rho) s_{\mu^t}(q^{\rho+\gamma}) q^{\kappa_\mu/2} = \sum_\gamma c^\gamma_{\alpha\beta} s_\gamma(q^{\rho+\mu^t}) s_{\mu^t}(q^\rho) q^{\kappa_\mu/2} \\
&= s_\alpha(q^{\rho+\mu^t}) s_\beta(q^{\rho+\mu^t}) s_{\mu^t}(q^\rho) q^{\kappa_\mu/2} = s_\alpha(q^\rho) s_\beta(q^\rho) s_{\mu^t}(q^{\rho+\alpha}) \frac{s_{\mu^t}(q^{\rho+\beta})}{s_{\mu^t}(q^\rho)} q^{\kappa_\mu/2}.
\end{aligned}
\tag{8.30}
$$

Applying this expression to (8.20), we find that the partition function in this case coincides with (8.27):

$$
Z^{\text{open }(3)}_{\text{BPS }1}(X, Q_b, Q_{fL}, Q_{fR}; q) = Z^{\text{open }(2)}_{\text{BPS }1}(X, Q_b, Q_{fL}, Q_{fR}; q). \tag{8.31}
$$

Summing these three contributions, we find the partition function in one instanton

sector.

$$
\begin{aligned}
&Z^{\rm open}_{\rm BPS\ 1}(X, Q_b, Q_{fL}, Q_{fR}; q)\\
&= Q_b \frac{q^{1/2}}{q-1}\frac{1}{1-Q_{fL}} \sum_{\mathfrak{m}'=0}^{\infty} \frac{X^{-1}(Xq^{1/2})^{\mathfrak{m}'}}{\prod_{k=1}^{\mathfrak{m}'}(1-q^k)(1-Q_{fR}q^{k-1})}\\
&\quad +2Q_b \frac{q}{(q-1)^2}\frac{1}{1-Q_{fL}} \sum_{\mathfrak{m}=0}^{\infty} \frac{1}{(1-Q_{fR}q^{-1})(1-Q_{fR}q^{\mathfrak{m}})} \frac{(Xq^{1/2})^{\mathfrak{m}}}{\prod_{k=1}^{\mathfrak{m}}(1-q^k)\prod_{k=1}^{\mathfrak{m}-1}(1-Q_{fR}q^{k-1})}.
\end{aligned}
\tag{8.32}
$$

In the four dimensional limit $\beta \to 0$, the open BPS partition function become

$$
\begin{aligned}
&Z^{\rm open(4D)}_{\rm BPS\ 1}(w, \Lambda, a_L, a_R; \hbar)\\
&= -\Lambda^4 \sum_{\mathfrak{m}'=0}^{\infty} (\Lambda^2 w)^{\mathfrak{m}'-1} \frac{1}{(2a_L)\hbar^{\mathfrak{m}'+1}\mathfrak{m}'! \prod_{k=1}^{\mathfrak{m}'}(2a_R+(k-1)\hbar)}\\
&\quad +2\Lambda^4 \sum_{\mathfrak{m}=1}^{\infty} (\Lambda^2 w)^{\mathfrak{m}} \frac{1}{(2a_L)(2a_R-\hbar)(2a_R+\mathfrak{m}\hbar)\hbar^{\mathfrak{m}+2}\mathfrak{m}! \prod_{k=1}^{\mathfrak{m}-1}(2a_R+(k-1)\hbar)},
\end{aligned}
\tag{8.33}
$$

where $q = e^{-\beta\hbar}$.

On the other hand, in the self-dual case $\epsilon_1 = -\epsilon_2 = \hbar$, the one instanton partition function for the gauge theory (8.17) yields

$$
\begin{aligned}
&Z^{(4D)}_{\text{1-inst}}(a, \hbar, -\hbar; \Lambda^2 w, \Lambda^2 w^{-1})\\
&= -\Lambda^4 \sum_{\mathfrak{m}=-1}^{\infty} (\Lambda^2 w)^{\mathfrak{m}} \frac{1}{\hbar^{\mathfrak{m}+2}(\mathfrak{m}+1)! \prod_{k=0}^{\mathfrak{m}+1}(2a+(k-1)\hbar)}\\
&\quad +2\Lambda^4 \sum_{\mathfrak{m}=0}^{\infty} (\Lambda^2 w)^{\mathfrak{m}} \frac{1}{(2a-\hbar)^2(2a+\mathfrak{m}\hbar)\hbar^{\mathfrak{m}+2}\mathfrak{m}! \prod_{k=1}^{\mathfrak{m}-1}(2a+(k-1)\hbar)}.
\end{aligned}
\tag{8.34}
$$

Choosing a_L and a_R by

$$
a_L = a - \hbar/2, \quad a_R = a, \tag{8.35}
$$

we find a coincidence between the one instanton partition function for gauge theory and four dimensional limit of the partition function for the open BPS states in the A model.

8.2.1 Geometric engineering of $N_f = 1$ theory

Geometrically the four dimensional gauge theory with $N_f = 1$ flavor is engineered by the A model on local del Pezzo surface dP_2 with Kähler parameters

$$
Q_f = e^{-2\beta a}, \quad Q_e = e^{-\beta(a-m)}, \quad Q_b = 2\beta^3\Lambda^3. \tag{8.36}
$$

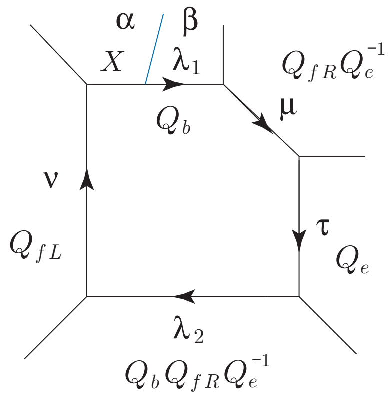

Figure 6: D-brane inserted on an inner leg of local del Pezzo surface

So as to realize the surface operator, we introduce toric D-brane as Fig.6, and the open string moduli is also identified by

$$X = \beta^2\Lambda^2 w. \tag{8.37}$$

The topological vertex computes the open BPS invariants on local del Pezzo surface.

$$Z_{\text{BPS}}^{\text{open}}(X, Q_b, Q_e, Q_{fL}, Q_{fR}; q) = \frac{Z^{\text{D-brane}}(X, Q_b, Q_e, Q_{fL}, Q_{fR}; q)}{Z_{\text{BPS}}^{\text{closed}}(Q_b, Q_e, Q_{fL}, Q_{fR}; q)}, \tag{8.38}$$

$$\begin{aligned} &Z_{\text{BPS}}^{\text{closed}}(Q_b, Q_e, Q_{fL}, Q_{fR}; q) \\ &= \sum_{\lambda_1,\lambda_2,\mu,\nu,\tau} C_{\lambda_1\nu^t\emptyset}C_{\mu\lambda_1^t\emptyset}C_{\tau\mu^t\emptyset}C_{\lambda_2\tau^t\phi}C_{\nu\lambda_2^t\emptyset} \\ &\qquad \times q^{(\kappa_\nu+\kappa_{\lambda_2})/2}(-Q_b)^{|\lambda_1|}(-Q_{fR}Q_e^{-1})^{|\mu|}(-Q_e)^{|\tau|}(Q_bQ_{fR}Q_e^{-1})^{|\lambda_2|}Q_{fL}^{|\nu|}, \end{aligned} \tag{8.39}$$

$$\begin{aligned} &Z_{\text{BPS}}^{\text{D-brane}}(X, Q_b, Q_e, Q_{fL}, Q_{fR}; q) \\ &= \sum_{\lambda_1,\lambda_2,\mu,\nu,\tau,\alpha,\beta} C_{(\lambda_1\otimes\alpha)\nu^t\emptyset}C_{\mu(\lambda_1^t\otimes\beta)\emptyset}C_{\tau\mu^t\emptyset}C_{\lambda_2\tau^t\phi}C_{\nu\lambda_2^t\emptyset} \\ &\qquad \times q^{(p\kappa_{\lambda_1\otimes\alpha}+p\kappa_{\lambda_1^t\otimes\beta}+\kappa_\nu+\kappa_{\lambda_2})/2}(-Q_b)^{|\lambda_1|}(-Q_{fR}Q_e^{-1})^{|\mu|}(-Q_e)^{|\tau|}(Q_bQ_{fR}Q_e^{-1})^{|\lambda_2|}Q_{fL}^{|\nu|}Q_b^{|\beta|} \\ &\qquad \times(-1)^{p|\alpha|+p|\beta|}\ \mathrm{Tr}_\alpha V\ \mathrm{Tr}_\beta V^{-1}. \end{aligned} \tag{8.40}$$

For later convenience, we have changed the Kähler parameter Q_f as in the local Hirzebruch case.

$$Q_{fL} = e^{-2\beta a_L}, \quad Q_{fR} = e^{-2\beta a_R}, \quad Q_e = e^{-\beta(a_R-m)}. \tag{8.41}$$

In the following we choose the framing $p = -1$.

The one instanton sector for four dimensional theory comes from a part of the above representation sums which satisfies $|\beta|+|\lambda_1|+|\lambda_2|=1$ for the D-brane partition function and $|\lambda_1|+|\lambda_2|=0$ for closed string partition function. We find the closed string partition function

$$Z^{\text{closed}}_{\text{BPS }0}(Q_b,Q_e,Q_{fL},Q_{fR};q)=\frac{M(Q_{fL};q)M(Q_{fR};q)}{M(Q_e;q)M(Q_{fR}Q_e^{-1};q)}. \tag{8.42}$$

The computation of the D-brane partition function for the one instanton sector is classified into three cases (8.23). Each partition function is computed in the same way as local Hirzebruch surface.

$$\begin{aligned}
&Z^{\text{open (1)}}_{\text{BPS }1}(X,Q_b,Q_e,Q_{fL},Q_{fR};q)\\
&=(-Q_b)\frac{q^{1/2}}{q-1}\sum_{\mathfrak{m}'=0}^{\infty}\frac{(1-Q_{fR}Q_e^{-1})X^{-1}(Xq^{1/2})^{\mathfrak{m}'}}{(1-Q_{fR})\prod_{k=1}^{\mathfrak{m}'}(1-q^k)(1-Q_{fL}q^{k-1})}, \qquad (8.43)\\
&Z^{\text{open (2)}}_{\text{BPS }1}(X,Q_b,Q_e,Q_{fL},Q_{fR};q)\\
&=(Q_bQ_fQ_e^{-1})\left(\frac{q^{1/2}}{q-1}\right)^2\\
&\times\sum_{\mathfrak{m}=0}^{\infty}\frac{(1-Q_e)(Xq^{1/2})^{\mathfrak{m}}}{(1-Q_{fR})(1-Q_{fL}q^{-1})(1-Q_{fL}q^{\mathfrak{m}})\prod_{k=1}^{\mathfrak{m}}(1-q^k)\prod_{k=1}^{\mathfrak{m}-1}(1-Q_{fL}q^{k-1})}, \qquad (8.44)\\
&Z^{\text{open (3)}}_{\text{BPS }1}(X,Q_b,Q_e,Q_{fL},Q_{fR};q)\\
&=(-Q_b)\left(\frac{q^{1/2}}{q-1}\right)^2\\
&\times\sum_{\mathfrak{m}=0}^{\infty}\frac{(1-Q_{fR}Q_e^{-1})(Xq^{1/2})^{\mathfrak{m}}}{(1-Q_{fR})(1-Q_{fL}q^{-1})(1-Q_{fL}q^{\mathfrak{m}})\prod_{k=1}^{\mathfrak{m}}(1-q^k)\prod_{k=1}^{\mathfrak{m}-1}(1-Q_{fL}q^{k-1})}. \qquad (8.45)
\end{aligned}$$

Summing all these contributions, one finds

$$\begin{aligned}
&Z^{\text{open}}_{\text{BPS }1}(X,Q_b,Q_e,Q_{fL},Q_{fR};q)\\
&=Q_b\frac{q^{1/2}}{q-1}\sum_{\mathfrak{m}'=0}^{\infty}\frac{(Q_{fR}Q_e^{-1}-1)X^{-1}(Xq^{1/2})}{(1-Q_{fR})\prod_{k=1}^{\mathfrak{m}'}(1-q^k)(1-Q_{fL}q^{k-1})}\\
&+Q_b\left(\frac{q^{1/2}}{q-1}\right)^2\sum_{\mathfrak{m}=0}^{\infty}\frac{(2Q_{fR}Q_e^{-1}-Q_{fR}-1)(Xq^{1/2})^{\mathfrak{m}}}{(1-Q_{fR})(1-Q_{fL}q^{-1})(1-Q_{fL}q^{\mathfrak{m}})\prod_{k=1}^{\mathfrak{m}}(1-q^k)\prod_{k=1}^{\mathfrak{m}-1}(1-Q_{fL}q^{k-1})}.
\end{aligned} \tag{8.46}$$

In the four dimensional limit ($\beta \to 0$), this partition function yields

$$\begin{aligned}
&Z^{\text{open(4D)}}_{\text{BPS } 1}(w, \Lambda, a_L, a_R, M; \hbar) \\
&= 2\Lambda^3 \sum_{\mathfrak{m}'=0}^{\infty} (\Lambda^2 w)^{\mathfrak{m}'-1} \frac{a_R + m}{(2a_R)\hbar^{\mathfrak{m}'+1}\mathfrak{m}'! \prod_{k=1}^{\mathfrak{m}'}(2a_L + (k-1)\hbar)} \\
&\quad + 2\Lambda^3 \sum_{\mathfrak{m}=1}^{\infty} (\Lambda^2 w)^{\mathfrak{m}} \frac{-2m}{(2a_R)(2a_L - \hbar)(2a_L + \mathfrak{m}\hbar)\hbar^{\mathfrak{m}+2}\mathfrak{m}! \prod_{k=1}^{\mathfrak{m}-1}(2a_L + (k-1)\hbar)}.
\end{aligned} \tag{8.47}$$

In the self-dual case $\epsilon_1 = -\epsilon_2 = \hbar$, the one instanton partition function (8.15) for the gauge theory is reduced to

$$\begin{aligned}
&Z^{(4\text{D})}_{1\text{-inst}}(a, \hbar, -\hbar, M_1; \Lambda^2 w, 2\Lambda^3 w^{-1}) \\
&= 2\Lambda^3 \sum_{\mathfrak{m}=-1}^{\infty} (\Lambda^2 w)^{\mathfrak{m}} \frac{a + 2M_1 - \hbar}{(2a - \hbar)(2a + \mathfrak{m}\hbar)\hbar^{\mathfrak{m}+2}(\mathfrak{m}+1)! \prod_{k=1}^{\mathfrak{m}}(2a + k\hbar)} \\
&\quad + 2\Lambda^3 \sum_{\mathfrak{m}=0}^{\infty} (\Lambda^2 w)^{\mathfrak{m}} \frac{-(4M_1 - \hbar)}{(2a - \hbar)^2(2a + \mathfrak{m}\hbar)\hbar^{\mathfrak{m}+2}\mathfrak{m}! \prod_{k=1}^{\mathfrak{m}-1}(2a + (k-1)\hbar)}.
\end{aligned} \tag{8.48}$$

This result coincides with the topological vertex computation (8.47) under the following shifts of parameters:

$$a_L = a, \quad a_R = a - \hbar/2, \quad m = 2M_1 - \hbar/2. \tag{8.49}$$

Acknowledgments

We would like to thank S. Hirano, H. Itoyama, H. Nagoya and S.Yanagida for enlightening discussions. We also thank T. Dimofte, M. Mariño and Y. Tachikawa for helpful correspondences. Special thanks are due to K. Nagao for his clear explanation of the work of Braverman and Etingof. This work is partially supported by the Grant-in-Aid for Nagoya University Global COE Program, "Quest for Fundamental Principles in the Universe: from Particles to the Solar System and the Cosmos", from the Ministry of Education, Culture, Sports, Science and Technology of Japan. The work of H.A and H.K. is supported in part by Daiko Foundation. The work is also supported in part by Grant-in-Aid for Young Scientists (B) [# 21740179] (H.F.), Grant-in-Aid for Scientific Research [# 22224001] (H.K.) and Grant-in-Aid for Scientific Research [#21340036] (Y.Y.) from the Japan Ministry of Education, Culture, Sports, Science and Technology.

Note added

After this paper was submitted to arXiv, there appeared a new article [61], where the results in [7] are extended to affine $sl(N)$ case.

Appendix A : Equivariant Character of the Affine Laumon space

The fixed points of the toric action on the affine $U(2)$ Laumon space are isolated and labeled by a pair of partitions $\vec{\lambda} := (\lambda_1, \lambda_2)$. We denote by $\lambda_{k,i}$ the i-th component of the partition $\lambda_k = (\lambda_{k,1}, \lambda_{k,2}, \cdots, \lambda_{k,N})$. The equivariant character $\mathrm{ch}_{\vec{\lambda},\vec{\mu}}(\vec{a}, \vec{b}) := \mathrm{Tr}_{\mathrm{Ext}(\vec{\lambda},\vec{\mu})}[\mathrm{diag.}(\epsilon_1, \epsilon_2; \vec{a}, \vec{b})]$ in [11] computes the contribution of a bifundamental multiplet, from which those of an adjoint and an (anti-)fundamental multiplet are derived. Hence the relevant gauge group is $U(2) \times U(2)$ in the following. We need a second pair of partitions $\vec{\mu} := (\mu_1, \mu_2)$ and the Coulomb moduli parameters $\vec{a} := (a_1, a_2), \vec{b} := (b_1, b_2)$ to write down the formula of the equivariant character. With the convention $a_k \equiv a_{k+2}$ and $\lambda_{k,i} \equiv \lambda_{k+2,i}$, the equivariant character at a fixed point of the toric action is[10]

$$
\begin{aligned}
\mathrm{ch}_{\vec{\lambda},\vec{\mu}}(\vec{a}, \vec{b}) := &\sum_{k,\ell \geq 1} e^{a_k - b_{\ell+1}} e^{\epsilon_1 + \epsilon_2(\lfloor \frac{\ell}{2} - \frac{1}{2} \rfloor - \lfloor \frac{k}{2} - 1 \rfloor)} \frac{(e^{\epsilon_1 \mu_{\ell+1,\ell}} - 1)(e^{-\epsilon_1 \lambda_{k,k}} - 1)}{e^{\epsilon_1} - 1} \\
&+ \sum_{k,\ell \geq 1} e^{a_{k+1} - b_\ell} e^{\epsilon_1 + \epsilon_2(\lfloor \frac{\ell}{2} - 1 \rfloor - \lfloor \frac{k}{2} - \frac{3}{2} \rfloor)} \frac{(e^{\epsilon_1 \mu_{\ell,\ell}} - 1)(e^{-\epsilon_1 \lambda_{k+1,k}} - 1)}{e^{\epsilon_1} - 1} \\
&+ \sum_{\ell \geq 1} e^{a_1 - b_{\ell+1}} e^{\epsilon_1 + \epsilon_2(\lfloor \frac{\ell}{2} - \frac{1}{2} \rfloor + 1)} \frac{e^{\epsilon_1 \mu_{\ell+1,\ell}} - 1}{e^{\epsilon_1} - 1} \\
&+ \sum_{\ell \geq 1} e^{a_2 - b_\ell} e^{\epsilon_1 + \epsilon_2(\lfloor \frac{\ell}{2} - 1 \rfloor + 1)} \frac{e^{\epsilon_1 \mu_{\ell,\ell}} - 1}{e^{\epsilon_1} - 1} \\
&- \sum_{k,\ell \geq 1} e^{a_k - b_\ell} e^{\epsilon_1 + \epsilon_2(\lfloor \frac{\ell}{2} - 1 \rfloor - \lfloor \frac{k}{2} - 1 \rfloor)} \frac{(e^{\epsilon_1 \mu_{\ell,\ell}} - 1)(e^{-\epsilon_1 \lambda_{k,k}} - 1)}{e^{\epsilon_1} - 1} \\
&- \sum_{k,\ell \geq 1} e^{a_{k+1} - b_{\ell+1}} e^{\epsilon_1 + \epsilon_2(\lfloor \frac{\ell}{2} - \frac{3}{2} \rfloor - \lfloor \frac{k}{2} - \frac{3}{2} \rfloor)} \frac{(e^{\epsilon_1 \mu_{\ell+1,\ell}} - 1)(e^{-\epsilon_1 \lambda_{k+1,k}} - 1)}{e^{\epsilon_1} - 1} \\
&- \sum_{k \geq 1} e^{a_k - b_1} e^{\epsilon_1 + \epsilon_2(-1 - \lfloor \frac{k}{2} - 1 \rfloor)} \frac{e^{-\epsilon_1 \lambda_{k,k}} - 1}{e^{\epsilon_1} - 1} \\
&- \sum_{k \geq 1} e^{a_{k+1} - b_2} e^{\epsilon_1 + \epsilon_2(-1 - \lfloor \frac{k}{2} - \frac{3}{2} \rfloor)} \frac{e^{-\epsilon_1 \lambda_{k+1,k}} - 1}{e^{\epsilon_1} - 1}.
\end{aligned}
\tag{A.1}
$$

Here the floor function $\lfloor k \rfloor$ denotes the largest integer not greater than k. We have rewritten the original formula by Feigin et. al. ([11]. Prop.4.15) to arrive at (A.1).

We can rewrite this character as a Laurent polynomial in e^{a_i}, e^{b_i} and e^{ϵ_i} with non-negative integer coefficients as follows:

[10] In the $SU(2)$ case, $a_k := (-1)^{k-1} a$ and $b_k := (-1)^{k-1} b$.

Proposition.

$$
\begin{aligned}
\mathrm{ch}_{\vec{\lambda},\vec{\mu}}(\vec{a},\vec{b}) = &\sum_{\substack{k\geq 1\\ \ell\geq 0}} e^{a_k-b_{\ell+1}} e^{\epsilon_2(\lfloor\frac{\ell+1}{2}\rfloor-\lfloor\frac{k}{2}\rfloor)} \sum_{i=\min(1,1+\mu_{\ell+1,\ell+1}-\lambda_{k,k})}^{\min(0,\mu_{\ell+1,\ell}-\lambda_{k,k})} e^{i\epsilon_1} \\
&+ \sum_{\substack{k\geq 1\\ \ell\geq 0}} e^{a_{k-1}-b_\ell} e^{\epsilon_2(\lfloor\frac{\ell}{2}\rfloor-\lfloor\frac{k-1}{2}\rfloor)} \sum_{i=\min(1,1+\mu_{\ell,\ell+1}-\lambda_{k-1,k})}^{\min(0,\mu_{\ell,\ell}-\lambda_{k-1,k})} e^{i\epsilon_1} \\
&+ \sum_{\substack{k\geq 0\\ \ell\geq 1}} e^{a_k-b_\ell} e^{\epsilon_2(\lfloor\frac{\ell}{2}\rfloor-\lfloor\frac{k}{2}\rfloor)} \sum_{i=\max(1,1+\mu_{\ell,\ell}-\lambda_{k,k})}^{\max(0,\mu_{\ell,\ell}-\lambda_{k,k+1})} e^{i\epsilon_1} \\
&+ \sum_{\substack{k\geq 0\\ \ell\geq 1}} e^{a_{k+1}-b_{\ell+1}} e^{\epsilon_2(\lfloor\frac{\ell+1}{2}\rfloor-\lfloor\frac{k+1}{2}\rfloor)} \sum_{i=\max(1,1+\mu_{\ell+1,\ell}-\lambda_{k+1,k})}^{\max(0,\mu_{\ell+1,\ell}-\lambda_{k+1,k+1})} e^{i\epsilon_1}
\end{aligned}
\tag{A.2}
$$

with $\lambda_{k,0} = \mu_{k,0} := \infty$.

Proof. Since

$$
q\frac{(q^N-1)(q^{-M}-1)}{q-1} = \left[\sum_{i=1-M}^{\min(0,N-M)} - \sum_{i=\max(1,1+N-M)}^{N}\right] q^i
\tag{A.3}
$$

for any $M, N = 0, 1, 2, 3, \cdots$, the character $\mathrm{ch}_{\vec{\lambda},\vec{\mu}}(\vec{a},\vec{b})$ reduces to

$$
\sum_{k,\ell\geq 1} e^{a_k-b_{\ell+1}} e^{\epsilon_2(\lfloor\frac{\ell+1}{2}\rfloor-\lfloor\frac{k}{2}\rfloor)} \left[\sum_{i=1-\lambda_{k,k}}^{\min(0,\mu_{\ell+1,\ell}-\lambda_{k,k})} - \sum_{i=\max(1,1+\mu_{\ell+1,\ell}-\lambda_{k,k})}^{\mu_{\ell+1,\ell}}\right] e^{i\epsilon_1}
\tag{A.4}
$$

$$
+ \sum_{k,\ell\geq 1} e^{a_{k-1}-b_\ell} e^{\epsilon_2(\lfloor\frac{\ell}{2}\rfloor-\lfloor\frac{k-1}{2}\rfloor)} \left[\sum_{i=1-\lambda_{k-1,k}}^{\min(0,\mu_{\ell,\ell}-\lambda_{k-1,k})} - \sum_{i=\max(1,1+\mu_{\ell,\ell}-\lambda_{k-1,k})}^{\mu_{\ell,\ell}}\right] e^{i\epsilon_1}
\tag{A.5}
$$

$$
+ \sum_{\ell\geq 1} e^{a_1-b_{\ell+1}} e^{\epsilon_2\lfloor\frac{\ell+1}{2}\rfloor} \sum_{i=1}^{\mu_{\ell+1,\ell}} e^{i\epsilon_1} + \sum_{\ell\geq 1} e^{a_0-b_\ell} e^{\epsilon_2\lfloor\frac{\ell}{2}\rfloor} \sum_{i=1}^{\mu_{\ell,\ell}} e^{i\epsilon_1}
\tag{A.6}
$$

$$
+ \sum_{k,\ell\geq 1} e^{a_k-b_\ell} e^{\epsilon_2(\lfloor\frac{\ell}{2}\rfloor-\lfloor\frac{k}{2}\rfloor)} \left[\sum_{i=\max(1,1+\mu_{\ell,\ell}-\lambda_{k,k})}^{\mu_{\ell,\ell}} - \sum_{i=1-\lambda_{k,k}}^{\min(0,\mu_{\ell,\ell}-\lambda_{k,k})}\right] e^{i\epsilon_1}
\tag{A.7}
$$

$$
+ \sum_{k,\ell\geq 1} e^{a_{k+1}-b_{\ell+1}} e^{\epsilon_2(\lfloor\frac{\ell+1}{2}\rfloor-\lfloor\frac{k+1}{2}\rfloor)} \left[\sum_{i=\max(1,1+\mu_{\ell+1,\ell}-\lambda_{k+1,k})}^{\mu_{\ell+1,\ell}} - \sum_{i=1-\lambda_{k+1,k}}^{\min(0,\mu_{\ell+1,\ell}-\lambda_{k+1,k})}\right] e^{i\epsilon_1}
\tag{A.8}
$$

$$
+ \sum_{k\geq 1} e^{a_k-b_1} e^{-\epsilon_2\lfloor\frac{k}{2}\rfloor} \sum_{i=1-\lambda_{k,k}}^{0} e^{i\epsilon_1} + \sum_{k\geq 1} e^{a_{k+1}-b_2} e^{\epsilon_2(1-\lfloor\frac{k+1}{2}\rfloor)} \sum_{i=1-\lambda_{k+1,k}}^{0} e^{i\epsilon_1}.
\tag{A.9}
$$

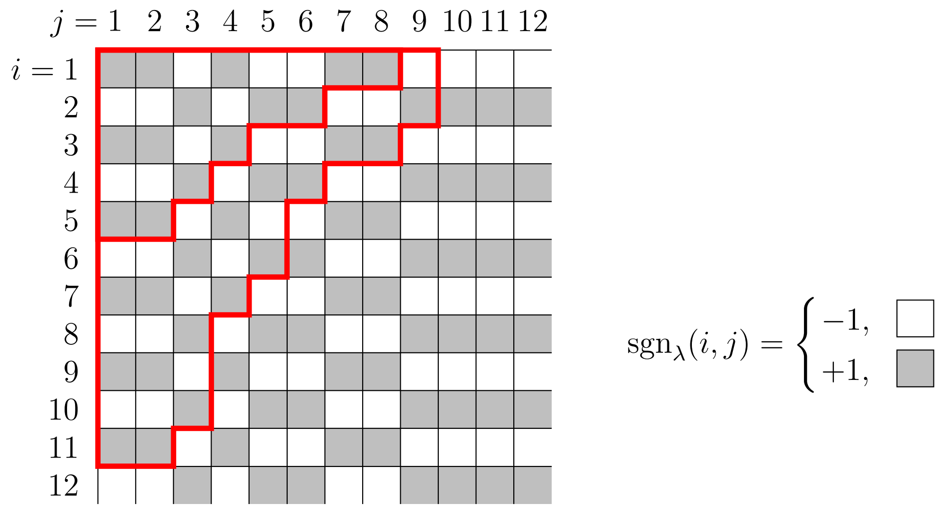

Figure 7: Example of $\mathrm{sgn}_\lambda(i,j)$ for $\lambda=(9,9,8,6,5,5,4,3,3,3,2)$, which equals that for $\lambda=(8,6,4,3,2)$. The black and white boxes denote $\mathrm{sgn}_\lambda(i,j)=1$ and -1, respectively.

Adding the second term of (A.7) with $\ell\geq 2$ and the first term of (A.4) yields

$$\sum_{k,\ell\geq 1} e^{a_k-b_{\ell+1}} e^{\epsilon_2(\lfloor\frac{\ell+1}{2}\rfloor-\lfloor\frac{k}{2}\rfloor)} \sum_{i=\min(1,1+\mu_{\ell+1,\ell+1}-\lambda_{k,k})}^{\min(0,\mu_{\ell+1,\ell}-\lambda_{k,k})} e^{i\epsilon_1}. \tag{A.10}$$

On the other hand, adding the second term of (A.7) with $\ell=1$ and the first term of (A.9) yields

$$\sum_{k\geq 1} e^{a_k-b_1} e^{-\epsilon_2\lfloor\frac{k}{2}\rfloor} \sum_{i=\min(1,1+\mu_{1,1}-\lambda_{k,k})}^{0} e^{i\epsilon_1}. \tag{A.11}$$

Combining (A.10) with (A.11) gives the first term of (A.2). In the same manner we can get other terms. □

Let us introduce the following signature (Fig. 7)

$$\mathrm{sgn}_\lambda(i,j) := \begin{cases} -1, & \text{if } \begin{cases} \lambda_{2n+1} < j \leq \lambda_{2n} & \text{and } \ i=2m-1 \ \text{ or} \\ \lambda_{2n+2} < j \leq \lambda_{2n+1} & \text{and } \ i=2m, \end{cases} \\ +1, & \text{if } \begin{cases} \lambda_{2n+1} < j \leq \lambda_{2n} & \text{and } \ i=2m \quad \text{ or} \\ \lambda_{2n+2} < j \leq \lambda_{2n+1} & \text{and } \ i=2m-1 \end{cases} \end{cases} \tag{A.12}$$

with $n=0,1,2,\cdots$ and $m=1,2,3,\cdots$. Here $\lambda_0:=\infty$. Note that if $\lambda_{k+1}=\lambda_{k+2}$ then $\mathrm{sgn}_\lambda(i,j)=\mathrm{sgn}_{\lambda^{\mathrm{red}}}(i,j)$ with $\lambda^{\mathrm{red}}:=(\lambda_1,\cdots,\lambda_k,\lambda_{k+3},\cdots)$. Then we can represent the character as a summation over some squares in the Young diagrams as follows:

Proposition.

$$
\begin{aligned}
\mathrm{ch}_{\vec{\lambda},\vec{\mu}}(\vec{a},\vec{b}) \; &= \; \sum_{I,J=1}^{2} \mathrm{ch}^{I,J}_{\lambda_I,\mu_J}(a_I,b_J), \\
\mathrm{ch}^{I,J}_{\lambda,\mu}(a,b) &:= \exp\left\{a-b+\frac{1}{2}\epsilon_1+\left(\frac{1}{2}+J-I\right)\frac{\epsilon_2}{2}\right\} \\
&\qquad \times \left\{ \sum_{\substack{(i,j)\in\lambda \\ \mathrm{sgn}_\mu(i,j)=(-1)^{I+J+1}}} \exp\left\{-\left(\lambda_i-j+\frac{1}{2}\right)\epsilon_1+\left(\mu'_j-i+\frac{1}{2}\right)\frac{\epsilon_2}{2}\right\}\right. \\
&\qquad\quad \left. + \sum_{\substack{(i,j)\in\mu \\ \mathrm{sgn}_\lambda(i,j)=(-1)^{I+J}}} \exp\left\{\quad\left(\mu_i-j+\frac{1}{2}\right)\epsilon_1-\left(\lambda'_j-i+\frac{1}{2}\right)\frac{\epsilon_2}{2}\right\}\right\}. \qquad \text{(A.13)}
\end{aligned}
$$

Moreover $\mathrm{ch}_{\vec{\lambda},\vec{\mu}}(\vec{a},\vec{b})$ *is symmetric under the replacement* $(a_1+\frac{\epsilon_2}{4},b_1+\frac{\epsilon_2}{4})\leftrightarrow(a_2-\frac{\epsilon_2}{4},b_2-\frac{\epsilon_2}{4})$ *and* $(\lambda_1,\mu_1)\leftrightarrow(\lambda_2,\mu_2)$.

Proof. Let $\mathrm{ch}^{I,J}_{\lambda_I,\mu_J}(a_I,b_J)$ be a part of $\mathrm{ch}_{\vec{\lambda},\vec{\mu}}(\vec{a},\vec{b})$, which contains $e^{a_I-b_J}$, i.e.,

$$
\begin{aligned}
\mathrm{ch}^{1,1}_{\lambda,\mu}(a,b) := &\sum_{\substack{k\geq 1\\ \ell\geq 0}} e^{a-b}e^{\epsilon_2(\ell-k+1)} \sum_{i=\min(1,1+\mu_{2\ell+1}-\lambda_{2k-1})}^{\min(0,\mu_{2\ell}-\lambda_{2k-1})} e^{i\epsilon_1} \\
&+ \sum_{k,\ell\geq 1} e^{a-b}e^{\epsilon_2(\ell-k)} \sum_{i=\min(1,1+\mu_{2\ell}-\lambda_{2k})}^{\min(0,\mu_{2\ell-1}-\lambda_{2k})} e^{i\epsilon_1} \\
&+ \sum_{k,\ell\geq 1} e^{a-b}e^{\epsilon_2(\ell-k)} \sum_{i=\max(1,1+\mu_{2\ell-1}-\lambda_{2k-1})}^{\max(0,\mu_{2\ell-1}-\lambda_{2k})} e^{i\epsilon_1} \\
&+ \sum_{\substack{k\geq 0\\ \ell\geq 1}} e^{a-b}e^{\epsilon_2(\ell-k)} \sum_{i=\max(1,1+\mu_{2\ell}-\lambda_{2k})}^{\max(0,\mu_{2\ell}-\lambda_{2k+1})} e^{i\epsilon_1}, \qquad \text{(A.14)}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{ch}^{2,1}_{\lambda,\mu}(a,b) := &\sum_{\substack{k\geq 1\\ \ell\geq 0}} e^{a-b}e^{\epsilon_2(\ell-k)} \sum_{i=\min(1,1+\mu_{2\ell+1}-\lambda_{2k})}^{\min(0,\mu_{2\ell}-\lambda_{2k})} e^{i\epsilon_1} \\
&+ \sum_{k,\ell\geq 1} e^{a-b}e^{\epsilon_2(\ell-k)} \sum_{i=\min(1,1+\mu_{2\ell}-\lambda_{2k-1})}^{\min(0,\mu_{2\ell-1}-\lambda_{2k-1})} e^{i\epsilon_1} \\
&+ \sum_{\substack{k\geq 0\\ \ell\geq 1}} e^{a-b}e^{\epsilon_2(\ell-k-1)} \sum_{i=\max(1,1+\mu_{2\ell-1}-\lambda_{2k})}^{\max(0,\mu_{2\ell-1}-\lambda_{2k+1})} e^{i\epsilon_1}
\end{aligned}
$$

$$+\sum_{k,\ell\geq 1} e^{a-b}e^{\epsilon_2(\ell-k)}\sum_{i=\max(1,1+\mu_{2\ell}-\lambda_{2k-1})}^{\max(0,\mu_{2\ell}-\lambda_{2k})} e^{i\epsilon_1} \tag{A.15}$$

and $\mathrm{ch}^{2,2}_{\lambda,\mu}(a,b) := \mathrm{ch}^{1,1}_{\lambda,\mu}(a,b)$ and $\mathrm{ch}^{1,2}_{\lambda,\mu}(a,b) := \mathrm{ch}^{2,1}_{\lambda,\mu}(a,b)e^{\epsilon_2}$. Then we obtain

$$\begin{aligned}
\mathrm{ch}^{1,1}_{\lambda,\mu}(a,b) = &\sum_{\substack{(i,j)\in\lambda\\ \mathrm{sgn}_\mu(i,j)=-1}} \exp\left\{a-b-(\lambda_i-j)\,\epsilon_1+\left(\mu'_j-i+1\right)\frac{\epsilon_2}{2}\right\}\\
&+\sum_{\substack{(i,j)\in\mu\\ \mathrm{sgn}_\lambda(i,j)=1}} \exp\left\{a-b+(\mu_i-j+1)\,\epsilon_1-\left(\lambda'_j-i\right)\frac{\epsilon_2}{2}\right\},\\
\mathrm{ch}^{2,1}_{\lambda,\mu}(a,b) = &\sum_{\substack{(i,j)\in\lambda\\ \mathrm{sgn}_\mu(i,j)=1}} \exp\left\{a-b-(\lambda_i-j)\,\epsilon_1+\left(\mu'_j-i\right)\frac{\epsilon_2}{2}\right\}\\
&+\sum_{\substack{(i,j)\in\mu\\ \mathrm{sgn}_\lambda(i,j)=-1}} \exp\left\{a-b+(\mu_i-j+1)\,\epsilon_1-\left(\lambda'_j-i+1\right)\frac{\epsilon_2}{2}\right\},
\end{aligned}\tag{A.16}$$

which proves (A.13). Since

$$\begin{aligned}
\mathrm{ch}^{2,2}_{\lambda_2,\mu_2}(a_2+\frac{\epsilon_2}{4},b_2+\frac{\epsilon_2}{4}) &= \mathrm{ch}^{1,1}_{\lambda_2,\mu_2}(a_2-\frac{\epsilon_2}{4},b_2-\frac{\epsilon_2}{4}),\\
\mathrm{ch}^{1,2}_{\lambda_1,\mu_2}(a_1-\frac{\epsilon_2}{4},b_2+\frac{\epsilon_2}{4}) &= \mathrm{ch}^{2,1}_{\lambda_1,\mu_2}(a_1+\frac{\epsilon_2}{4},b_2-\frac{\epsilon_2}{4}),
\end{aligned}\tag{A.17}$$

the proposition follows. □

Especially when $\vec{\mu}=\vec{\lambda}$, $\vec{\mu}=\vec{\emptyset}$ or $\vec{\lambda}=\vec{\emptyset}$, we can also represent the character as a summation over all squares in the Young diagrams:

Corollary.

$$\begin{aligned}
\mathrm{ch}_{\vec{\lambda},\vec{\lambda}}(\vec{a},\vec{b}) &= \sum_{I,J=1}^{2}\widetilde{\mathrm{ch}}^{I,J}_{\lambda_I,\lambda_J}(\vec{a},\vec{b}),\\
\widetilde{\mathrm{ch}}^{I,J}_{\lambda,\mu}(\vec{a},\vec{b}) &:= \sum_{(i,j)\in\lambda}\exp\left\{a_{I+(1-\mathrm{sgn}_\mu(i,j)^{I-J})/2}-b_{J+(1-\mathrm{sgn}_\mu(i,j)^{I-J})/2}+\frac{1}{2}\left(\epsilon_1+\frac{\epsilon_2}{2}\right)\right\}\\
&\times\exp\left\{\left(\left(\lambda_i-j+\frac{1}{2}\right)\epsilon_1-\left(\mu'_j-i+\frac{1}{2}+J-I\right)\frac{\epsilon_2}{2}\right)(-1)^{I+J}\mathrm{sgn}_\mu(i,j)\right\},
\end{aligned}\tag{A.18}$$

$$\mathrm{ch}_{\vec{\lambda},\vec{\emptyset}}(\vec{a},\vec{b}) = \sum_{I=1}^{2} e^{a_I}\left[e^{-b_I}\sum_{\substack{(i,j)\in\lambda_I\\ i:\mathrm{odd}}} + e^{-b_{I+1}+\frac{\epsilon_2}{2}(-1)^{I+1}}\sum_{\substack{(i,j)\in\lambda_I\\ i:\mathrm{even}}}\right]e^{(1-j)\epsilon_1+(1-i)\frac{\epsilon_2}{2}}, \tag{A.19}$$

$$\mathrm{ch}_{\vec{\emptyset},\vec{\mu}}(\vec{a},\vec{b}) = \sum_{J=1}^{2} e^{-b_J}\left[e^{a_J}\sum_{\substack{(i,j)\in\mu_J\\ i:\mathrm{even}}} + e^{a_{J+1}+\frac{\epsilon_2}{2}(-1)^{J}}\sum_{\substack{(i,j)\in\mu_J\\ i:\mathrm{odd}}}\right]e^{j\epsilon_1+i\frac{\epsilon_2}{2}}. \tag{A.20}$$

Proof. Let $\widetilde{\mathrm{ch}}^{I,I}_{\lambda,\lambda}(\vec{a},\vec{b}) := \mathrm{ch}^{I,I}_{\lambda,\lambda}(a_I,b_I)$. For $I \neq J$, let $\widetilde{\mathrm{ch}}^{I,J}_{\lambda,\mu}(\vec{a},\vec{b})$ be the combination of the terms of $\mathrm{ch}^{I,J}_{\lambda,\mu}(a_I,b_J)$ with negative powers in e^{ϵ_1} and those of $\mathrm{ch}^{J,I}_{\mu,\lambda}(a_J,b_I)$ with positive powers. Then we get (A.18). When $\vec{\mu}=\vec{\emptyset}$, since $\mathrm{sgn}_{\emptyset}(i,j)=(-1)^i$, if i is odd or even number, then $I=J$ or $I\neq J$, respectively. Thus $\mathrm{ch}^{1,1}_{\lambda,\emptyset}(a_1,b_1)+\mathrm{ch}^{1,2}_{\lambda,\emptyset}(a_1,b_2)$ and $\mathrm{ch}^{2,2}_{\lambda,\emptyset}(a_2,b_2)+\mathrm{ch}^{2,1}_{\lambda,\emptyset}(a_2,b_1)$ give the $I=1$ and 2 part of (A.19), respectively. On the other hand, when $\vec{\lambda}=\vec{\emptyset}$, if i is even or odd number, then $I=J$ or $I\neq J$, respectively, and in the same manner we obtain (A.20). □

Appendix B : Multi points insertion of degenerate operators

B.1 $N_f = 0$ case

In $N_f=0$ case, we put $\langle A| = \langle \Delta', \Lambda|$ and $|B\rangle = |\Delta,\Lambda\rangle$ in eq.(5.6), then we have

$$\begin{aligned}&\Big[\frac{\Lambda}{4}\partial_\Lambda + \frac{\Delta+\Delta'-Nh_{1,2}}{2} + \Lambda^2(z_1+\frac{1}{z_1}) + b^2 z_1^2\partial_{z_1}^2 - \frac{3}{2}z_1\partial_{z_1}\\ &+\sum_{j=2}^{N}\big\{(\frac{z_1 z_j}{z_1-z_j}-\frac{z_j}{2})\partial_{z_j} + \frac{z_1^2 h_{1,2}}{(z_1-z_j)^2}\big\}\Big]\Psi = 0.\end{aligned} \tag{B.1}$$

We set the dimensions of initial state as $\Delta=\Delta(a_0)$, then the dimensions of intermediate and final states $\Delta(a_i)$ are restricted by the fusion rule as $a_{i+1}=a_i\pm\dfrac{1}{2b}$ and $\Delta'=\Delta(a_N)$. For each choice of the intermediate channels (called fusion path), one has a series solution of the form

$$\Psi = \prod_{i=1}^{N} z_i^{\Delta(a_i)-\Delta(a_{i-1})-h_{1,2}} \prod_{1\le i<j\le N}(1-\frac{z_j}{z_i})^{-\frac{1}{2b^2}}Y(z), \quad Y(z)=\sum_{n=0}^{\infty}Y_n(z)\Lambda^{2n}. \tag{B.2}$$

• The case $N=2$: For the simplest fusion path $a_i = a + \frac{i}{2b}$, we have

$$Y_0 = 1, \quad Y_1 = \frac{1}{-b^2-2ab-1}(\frac{1}{z_2}+\frac{1}{z_1}) + \frac{1}{-b^2+2ab+1}(z_1+z_2),$$

$$Y_2 = c_1 + c_2(\frac{1}{z_2^2}+\frac{1}{z_1^2}) + c_3(\frac{z_1}{z_2}+\frac{z_2}{z_1}) + c_4 z_1 z_2 + \frac{c_5}{z_1 z_2} + c_6(z_1^2+z_2^2),$$
$$c_1 = \frac{2(b^2+1)}{(b^2-2ab-1)(b^2+2ab+1)}, \quad c_2 = \frac{1}{2(b^2+2ab+1)(2b^2+2ab+1)},$$
$$c_3 = -\frac{1}{(-b^2+2ab+1)(b^2+2ab+1)}, \quad c_4 = \frac{2(a-b)}{(2a-b)(-2b^2+2ab+1)(-b^2+2ab+1)},$$
$$c_5 = \frac{2(b^2+ab+1)}{(b^2+2ab+1)(b^2+2ab+2)(2b^2+2ab+1)}, \quad c_6 = \frac{1}{2(-2b^2+2ab+1)(-b^2+2ab+1)},$$

$$\begin{aligned}
Y_3 &= c_1(\frac{1}{z_2^3}+\frac{1}{z_1^3}) + c_2(\frac{1}{z_2}+\frac{1}{z_1}) + c_3(z_1+z_2) + c_4(z_1^3+z_2^3) \\
&+c_5(\frac{1}{z_2^2 z_1}+\frac{1}{z_2 z_1^2}) + c_6(z_2 z_1^2 + z_2^2 z_1) + c_7(\frac{z_1}{z_2^2}+\frac{z_2}{z_1^2}) + c_8(\frac{z_1^2}{z_2}+\frac{z_2^2}{z_1}),
\end{aligned}$$
$$c_1 = -\frac{1}{6(b^2+2ab+1)(2b^2+2ab+1)(3b^2+2ab+1)}, \quad c_2 = \frac{8b^4+8ab^3+13b^2+6ab+6}{2(-b^2+2ab+1)(b^2+2ab+1)(b^2+2ab+2)(2b^2+2ab+1)},$$
$$c_3 = -\frac{-8b^3+8ab^2-5b+6a}{2(2a-b)(-2b^2+2ab+1)(-b^2+2ab+1)(b^2+2ab+1)}, \quad c_4 = \frac{1}{6(-3b^2+2ab+1)(-2b^2+2ab+1)(-b^2+2ab+1)},$$
$$c_5 = -\frac{3b^2+2ab+2}{2(b^2+2ab+1)(b^2+2ab+2)(2b^2+2ab+1)(3b^2+2ab+1)}, \quad c_6 = \frac{2a-3b}{2(2a-b)(-3b^2+2ab+1)(-2b^2+2ab+1)(-b^2+2ab+1)},$$
$$c_7 = \frac{1}{2(-b^2+2ab+1)(b^2+2ab+1)(2b^2+2ab+1)}, \quad c_8 = -\frac{1}{2(-2b^2+2ab+1)(-b^2+2ab+1)(b^2+2ab+1)}. \tag{B.3}$$

Then the free energy is given as

$$\log Y(z_1,z_2) = g(z_1) + g(z_2) + g(z_1,z_2), \tag{B.4}$$

where

$$\begin{aligned}
g(z_1) &= \Lambda^2(\tfrac{z_1}{2ab-b^2+1} - \tfrac{1}{z_1(2ab+b^2+1)}) \\
&+\Lambda^4(\tfrac{b^2 z_1^2}{2(2ab-2b^2+1)(2ab-b^2+1)^2} - \tfrac{b^2}{2z_1^2(2ab+b^2+1)^2(2ab+2b^2+1)} - \tfrac{b^2}{(2ab-b^2+1)(2ab+b^2+1)}) \\
&+\Lambda^6(\tfrac{2b^4 z_1^3}{3(2ab-3b^2+1)(2ab-2b^2+1)(2ab-b^2+1)^3} - \tfrac{2b^4}{z_1(2ab-b^2+1)(2ab+b^2+1)^2(2ab+b^2+2)(2ab+2b^2+1)} \\
&-\tfrac{2b^4}{3z_1^3(2ab+b^2+1)^3(2ab+2b^2+1)(2ab+3b^2+1)} + \tfrac{2b^3 z_1}{(2a-b)(2ab-2b^2+1)(2ab-b^2+1)^2(2ab+b^2+1)}) + \mathcal{O}(\Lambda^8),
\end{aligned} \tag{B.5}$$

and

$$\begin{aligned}
g(z_1,z_2) &= \Lambda^4(-\tfrac{bz_1 z_2}{(-2ab+b^2-1)^2(4a^2b-6ab^2+2a+2b^3-b)} - \tfrac{b^2}{z_1 z_2(2ab+b^2+1)^2(2ab+b^2+2)(2ab+2b^2+1)}) \\
&+\Lambda^6(-\tfrac{2b^4}{(2ab+b^2+1)^3(2ab+b^2+2)(2ab+2b^2+1)(2ab+3b^2+1)}(\tfrac{1}{z_1^2 z_2}+\tfrac{1}{z_1 z_2^2}) \\
&-\tfrac{2b^3(z_1 z_2^2 + z_1^2 z_2)}{(2a-b)(2ab-3b^2+1)(2ab-2b^2+1)(2ab-b^2+1)^3} + \mathcal{O}(\Lambda^8).
\end{aligned} \tag{B.6}$$

Under the limit $a \to \frac{a}{\hbar}$, $\Lambda \to \frac{\Lambda}{\hbar}$ $\hbar \to 0$, we have

$$-\frac{\Lambda^4(z_1^2 z_2^2+1)}{16a^4b^2 z_1 z_2} - \frac{\Lambda^6(z_1+z_2)(z_1^3 z_2^3+1)}{32a^6 b^2 z_1^2 z_2^2} + \mathcal{O}(\hbar). \tag{B.7}$$

This agrees with the B model results.

• The case $N=3$: For the simplest fusion path $a_i = a + \frac{i}{2b}$, we have

$$
\begin{aligned}
&Y_0(z)=1, \quad Y_1 = \frac{z_1+z_2+z_3}{2ab-b^2+2} - \frac{1}{(2ab+b^2+1)}\left(\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right),\\
&Y_2 = c_1 + c_2\left(\frac{1}{z_2^2}+\frac{1}{z_3^2}+\frac{1}{z_1^2}\right) + c_3(z_1^2+z_2^2+z_3^2) + c_4\left(\frac{z_1}{z_2}+\frac{z_1}{z_3}+\frac{z_3}{z_2}+\frac{z_2}{z_3}+\frac{z_2}{z_1}+\frac{z_3}{z_1}\right),\\
&+c_5(z_1z_2+z_3z_2+z_1z_3) + c_6\left(\frac{1}{z_1z_3}+\frac{1}{z_1z_2}+\frac{1}{z_3z_2}\right)\\
&c_1 = \frac{2b^2+3}{(-2ab+b^2-2)(2ab+b^2+1)}, \quad c_2 = \frac{1}{2(2ab+b^2+1)(2ab+2b^2+1)},\\
&c_3 = \frac{1}{4(ab-b^2+1)(2ab-b^2+2)}, \quad c_4 = -\frac{1}{(2ab-b^2+2)(2ab+b^2+1)},\\
&c_5 = \frac{2ab-2b^2+1}{2(ab-b^2+1)(2ab-b^2+1)(2ab-b^2+2)}, \quad c_6 = \frac{2(ab+b^2+1)}{(2ab+b^2+1)(2ab+b^2+2)(2ab+2b^2+1)},\\
&Y_3 = c_1\left(\frac{1}{z_2^3}+\frac{1}{z_3^3}+\frac{1}{z_1^3}\right) + c_2\left(\frac{1}{z_2}+\frac{1}{z_3}+\frac{1}{z_1}\right) + c_3(z_1+z_2+z_3)\\
&+c_4\left(z_1^3+z_2^3+z_3^3\right) + c_5\left(\frac{1}{z_2^2z_3}+\frac{1}{z_2z_3^2}+\frac{1}{z_2^2z_1}+\frac{1}{z_3^2z_1}+\frac{1}{z_2z_1^2}+\frac{1}{z_3z_1^2}\right)\\
&+c_6\left(z_2z_1^2+z_3z_1^2+z_2^2z_1+z_3^2z_1+z_2z_3^2+z_2^2z_3\right) + c_7\left(\frac{z_1}{z_2^2}+\frac{z_1}{z_3^2}+\frac{z_3}{z_2^2}+\frac{z_2}{z_3^2}+\frac{z_2}{z_1^2}+\frac{z_3}{z_1^2}\right)\\
&+c_8\left(\frac{z_1^2}{z_2}+\frac{z_1^2}{z_3}+\frac{z_3^2}{z_2}+\frac{z_2^2}{z_3}+\frac{z_2^2}{z_1}+\frac{z_3^2}{z_1}\right) + c_9\left(\frac{z_1}{z_2z_3}+\frac{z_3}{z_2z_1}+\frac{z_2}{z_3z_1}\right)\\
&+c_{10}\left(\frac{z_3z_2}{z_1}+\frac{z_1z_2}{z_3}+\frac{z_1z_3}{z_2}\right) + c_{11}\frac{1}{z_1z_2z_3} + c_{12}z_1z_2z_3,
\end{aligned}
\tag{B.8}
$$

where

$$
\begin{aligned}
c_1 &= -\frac{1}{6(2ab+b^2+1)(2ab+2b^2+1)(2ab+3b^2+1)},\\
c_2 &= -\frac{8ab^3+10ab+8b^4+17b^2+10}{2(-2ab+b^2-2)(2ab+b^2+1)(2ab+b^2+2)(2ab+2b^2+1)},\\
c_3 &= -\frac{8ab^3+10ab-8b^4-5b^2+5}{4(ab-b^2+1)(2ab-b^2+1)(2ab-b^2+2)(2ab+b^2+1)},\\
c_4 &= \frac{1}{12(2ab-3b^2+2)(ab-b^2+1)(2ab-b^2+2)},\\
c_5 &= -\frac{2ab+3b^2+2}{2(2ab+b^2+1)(2ab+b^2+2)(2ab+2b^2+1)(2ab+3b^2+1)},\\
c_6 &= \frac{2ab-3b^2+1}{4(2ab-3b^2+2)(ab-b^2+1)(2ab-b^2+1)(2ab-b^2+2)},\\
c_7 &= -\frac{1}{2(-2ab+b^2-2)(2ab+b^2+1)(2ab+2b^2+1)},\\
c_8 &= -\frac{1}{4(-2ab+b^2-2)(-ab+b^2-1)(2ab+b^2+1)},\\
c_9 &= -\frac{2(ab+b^2+1)}{(-2ab+b^2-2)(2ab+b^2+1)(2ab+b^2+2)(2ab+2b^2+1)},\\
c_{10} &= -\frac{2ab-2b^2+1}{2(ab-b^2+1)(2ab-b^2+1)(2ab-b^2+2)(2ab+b^2+1)},\\
c_{11} &= -\frac{2(2a^2b^2+5ab^3+5ab+3b^4+7b^2+3)}{(2ab+b^2+1)(2ab+b^2+2)(2ab+b^2+3)(2ab+2b^2+1)(2ab+3b^2+1)},\\
c_{12} &= \frac{4a^2b-10ab^2+2a+6b^3-b}{2(2a-b)(2ab-3b^2+2)(ab-b^2+1)(2ab-b^2+1)(2ab-b^2+2)}.
\end{aligned}
\tag{B.9}
$$

In the free energy $F = \log Y$, the relevant terms at order Λ^6 are

$$\frac{4b^3}{(2a-b)(2ab-3b^2+2)(ab-b^2+1)(2ab-b^2+1)(2ab-b^2+2)^3} z_1 z_2 z_3 \\ -\frac{8b^4}{(2ab+b^2+1)^3(2ab+b^2+2)(2ab+b^2+3)(2ab+2b^2+1)(2ab+3b^2+1)} \frac{1}{z_1 z_2 z_3}. \tag{B.10}$$

Under the limit $a \to \frac{a}{\hbar}$, $\hbar \to 0$, this gives

$$\hbar^7 \frac{1}{16a^7 b^3}(z_1 z_2 z_3 - \frac{1}{z_1 z_2 z_3}) + \mathcal{O}(\hbar^8). \tag{B.11}$$

This is consistent with the B model results.

B.2 $N_f = 1$ case

We put $\langle A| = \langle \Delta_-, \Lambda, m|$ and $|B\rangle = |\Delta_+, \Lambda\rangle$, then (5.6) takes the form

$$\Big[\Big(L_0\Big)_0 + \frac{\Lambda^2}{z_1} + \Big(b^2 z_1^2 \partial_{z_1}^2 - z_1 \partial_{z_1}\Big) + \sum_{j=2}^{N}\Big(\frac{z_1 z_j}{z_1 - z_j}\partial_{z_j} + \frac{z_1^2}{(z_1-z_j)^2} h_{1,2}\Big) \\ + z_1(-2m\Lambda) + z_1^2(-\Lambda^2)\Big]\Psi = 0, \tag{B.12}$$

where the action of the first term is given as

$$\Big(L_0\Big)_0 = \frac{\Lambda}{3}\partial_\Lambda + \frac{\Delta_- + 2\Delta_+}{3} - \frac{1}{3}\sum_{i=1}^{N}(z_i \partial_{z_i} + h_{1,2}), \tag{B.13}$$

by using the relations $L_0|B\rangle = (\Delta_+ + \frac{\Lambda}{2}\partial_\Lambda)|B\rangle$ and $\langle A|L_0 = (\Delta_- + \Lambda\partial_\Lambda)\langle A|$.

The equation has a solution such as $Y(z) = 1 + Y_1\Lambda + Y_2\Lambda^2 + \cdots$ with the same

pre-factor as $N_f = 0$ case. The first terms are as follows

$$Y_0 = 1, \quad Y_1 = -\frac{2m(z_1+z_2)}{-b^2+2ab+1},$$

$$Y_2 = -\frac{(-b^2+2ab-4m^2+1)(z_1^2+z_2^2)}{2(-2b^2+2ab+1)(-b^2+2ab+1)} + \frac{(-8m^2b^2-b^2+8am^2b+2ab+1)z_2z_1}{(2a-b)b(-2b^2+2ab+1)(-b^2+2ab+1)} - \frac{1}{(b^2+2ab+1)}\left(\frac{1}{z_1}+\frac{1}{z_2}\right),$$

$$\begin{aligned} Y_3 = &\frac{m(-5b^2+6ab-4m^2+3)(z_1^3+z_2^3)}{3(-3b^2+2ab+1)(-2b^2+2ab+1)(-b^2+2ab+1)} + \frac{m(3b^4-8ab^3+4a^2b^2+12m^2b^2+b^2-8am^2b-2ab-2)(z_1z_2^2+z_2z_1^2)}{(2a-b)b(-3b^2+2ab+1)(-2b^2+2ab+1)(-b^2+2ab+1)} \\ &+\frac{2mz_1}{(-b^2+2ab+1)(b^2+2ab+1)(z_1+z_2)} + \frac{4(b^2+1)m}{(-b^2+2ab+1)(b^2+2ab+1)}, \end{aligned}$$

$$\begin{aligned} Y_4 = &\frac{(9b^4-24ab^3+12a^2b^2+56m^2b^2-12b^2-48am^2b+12ab+16m^4-24m^2+3)(z_1^4+z_2^4)}{24(-4b^2+2ab+1)(-3b^2+2ab+1)(-2b^2+2ab+1)(-b^2+2ab+1)} \\ &-\frac{(80m^2b^4+9b^4-136am^2b^3-24ab^3+64m^4b^2+12a^2b^2+48a^2m^2b^2-12m^2b^2-12b^2-32am^4b+12ab-12m^2+3)(z_1z_2^3+z_2z_1^3)}{6(2a-b)b(-4b^2+2ab+1)(-3b^2+2ab+1)(-2b^2+2ab+1)(-b^2+2ab+1)} \\ &+\frac{C_1z_2^2z_1^2}{4(a-b)(2a-b)b^2(-4b^2+2ab+1)(-3b^2+2ab+1)(-2b^2+2ab+1)(-b^2+2ab+1)} \\ &+\frac{(-b^2+2ab-4m^2+1)z_2^2}{2(-2b^2+2ab+1)(-b^2+2ab+1)(b^2+2ab+1)}\left(\frac{1}{z_1}+\frac{1}{z_2}\right) \\ &+\frac{(32m^2b^4+5b^4-32am^2b^3-12ab^3+4a^2b^2+20m^2b^2-3b^2-24am^2b-2ab-2)(z_1+z_2)}{2(2a-b)b(-2b^2+2ab+1)(-b^2+2ab+1)(b^2+2ab+1)} \\ &+\frac{2(b^2+ab+1)}{(b^2+2ab+1)(b^2+2ab+2)(2b^2+2ab+1)z_2z_1} + \frac{1}{2(b^2+2ab+1)(2b^2+2ab+1)(z_1^2+z_2^2)}, \end{aligned} \tag{B.14}$$

$C_1 = -8bm^2(2a-3b)((2a^2b^2-5ab^3-ab+2b^4+b^2-1)+((2ab-3b^2+1)((2ab-b^2+1)((2a^2b^2-5ab^3+2b^4+1)+16b^2m^4(2a-3b)(a-2b)$.

Then the free energy $F = \log Y = g(z_1) + g(z_2) + g(z_1, z_2)$ is given by

$$\begin{aligned} g(z_1) = &-\frac{2mz_1L}{-b^2+2ab+1} + \left(-\frac{(-b^2+2ab-2mb+1)(-b^2+2ab+2mb+1)z_1^2}{2(-2b^2+2ab+1)(-b^2+2ab+1)^2} - \frac{1}{(b^2+2ab+1)z_1}\right)\Lambda^2 \\ &+\left(-\frac{4b^2m(b^2-2ab-2mb-1)(b^2-2ab+2mb-1)z_1^3}{3(b^2-2ab-1)^3(2b^2-2ab-1)(3b^2-2ab-1)} - \frac{2b^2m}{(b^2-2ab-1)(b^2+2ab+1)}\right)\Lambda^3 \\ &+\left(\frac{b^2(-b^2+2ab-2mb+1)(-b^2+2ab+2mb+1)C_2z_1^4}{4(-4b^2+2ab+1)(-3b^2+2ab+1)(-2b^2+2ab+1)^2(-b^2+2ab+1)^4}\right. \\ &\left.-\frac{2b(-b^2+2ab-2mb+1)(-b^2+2ab+2mb+1)z_1}{(2a-b)(-2b^2+2ab+1)(-b^2+2ab+1)^2(b^2+2ab+1)} - \frac{b^2}{2(b^2+2ab+1)^2(2b^2+2ab+1)z_1^2}\right)\Lambda^4 + \mathcal{O}(\Lambda^5), \end{aligned}$$

$$\begin{aligned} g(z_1,z_2) = &+\frac{(-b^2+2ab-2mb+1)(-b^2+2ab+2mb+1)z_1z_2\Lambda^2}{(2a-b)b(-2b^2+2ab+1)(-b^2+2ab+1)^2} - \frac{4bm(-b^2+2ab-2mb+1)(-b^2+2ab+2mb+1)z_1z_2(z_1+z_2)\Lambda^3}{(2a-b)(-3b^2+2ab+1)(-2b^2+2ab+1)(-b^2+2ab+1)^3} \\ &+\left(-\frac{b^2}{(b^2+2ab+1)^2(b^2+2ab+2)(2b^2+2ab+1)z_1z_2} - \frac{(-b^2+2ab-2mb+1)(-b^2+2ab+2mb+1)C_2z_1z_2(z_1^2+z_2^2)b}{(2a-b)(-4b^2+2ab+1)(-3b^2+2ab+1)(-2b^2+2ab+1)^2(-b^2+2ab+1)^4}\right. \\ &\left.-\frac{(-b^2+2ab-2mb+1)(-b^2+2ab+2mb+1)C_3z_1^2z_2^2}{4(a-b)(2a-b)^2(-4b^2+2ab+1)(-3b^2+2ab+1)(-2b^2+2ab+1)^2(-b^2+2ab+1)^4b}\right)\Lambda^4 + \mathcal{O}(\Lambda^5), \end{aligned} \tag{B.15}$$

$C_2 = -3b^6+14ab^5-20a^2b^4+44m^2b^4+7b^4+8a^3b^3-40am^2b^3-20ab^3+12a^2b^2-20m^2b^2-5b^2+6ab+1$,

$C_3 = -3b^{10}+26ab^9-88a^2b^8+172m^2b^8-11b^8+144a^3b^7-664am^2b^7+54ab^7-112a^4b^6-84a^2b^6+784a^2m^2b^6+20m^2b^6+40b^6+32a^5b^5+40a^3b^5-288a^3m^2b^5+24am^2b^5-150ab^5+168a^2b^4-32a^2m^2b^4-52m^2b^4-36b^4-56a^3b^3+64am^2b^3+82ab^3-44a^2b^2+4m^2b^2+11b^2-12ab-1$.

Under the limit $a \to \frac{a}{\hbar}$, $m \to \frac{m}{\hbar}$, $\Lambda \to \frac{\Lambda}{\hbar}$ and $\hbar \to 0$, we have

$$\begin{aligned} g(z_1, z_2) \to \frac{z_1 z_2\big(a^2 - m^2\big)}{4a^4 b^2}\Lambda^2 - \frac{m z_1 z_2 (z_1 + z_2)\big(a^2 - m^2\big)}{4a^6 b^2}\Lambda^3 - \Lambda^4 \Big(\frac{1}{16 a^4 b^2 z_1 z_2} \\ + \frac{z_1^2 z_2^2\big(a^2 - 9m^2\big)\big(a^2 - m^2\big)}{32 a^8 b^2} + \frac{z_1\big(z_1^2 + z_2^2\big) z_2\big(a^2 - 5m^2\big)\big(a^2 - m^2\big)}{16 a^8 b^2}\Big) + \mathcal{O}(\Lambda^5). \end{aligned} \tag{B.16}$$

Again, this recovers the B model results correctly.

Appendix C : Schur functions and topological vertex

The Schur function satisfies the following properties [60]:

$$s_\mu(cx) = c^{|\mu|} s_\mu(x), \quad s_\mu(q^\rho) = q^{\kappa_\mu/2} s_{\mu^t}(q^\rho), \quad s_\mu(q^\rho) = (-1)^{|\mu|} s_{\mu^t}(q^{-\rho}), \tag{C.1}$$

$$s_\mu(q^\rho) s_\nu(q^{\rho+\mu}) = s_\nu(q^\rho) s_\mu(q^{\rho+\nu}), \tag{C.2}$$

where $|\mu|$ and κ_μ are

$$|\mu| := \sum_i \mu_i, \tag{C.3}$$

$$\kappa_\mu := |\mu| + \sum_i \mu_i(\mu_i - 2i) = 2 \sum_{(i,j)\in\mu} (j - i), \quad \kappa_{\mu^t} = -\kappa_\mu. \tag{C.4}$$

The Cauchy formulas for the Schur functions are

$$\sum_\mu s_\mu(x) s_\mu(y) = \prod_{i,j} \frac{1}{1 - x_i y_j} = \exp\left[\sum_{n,i,j} \frac{1}{n} x_i^n y_j^n\right], \tag{C.5}$$

$$\sum_\mu s_\mu(x) s_{\mu^t}(y) = \prod_{i,j} (1 + x_i y_j) = \exp\left[-\sum_{n,i,j} \frac{(-1)^n}{n} x_i^n y_j^n\right]. \tag{C.6}$$

The topological vertex in the canonical framing is [30]

$$C_{\mu_1\mu_2\mu_3}(q) = q^{(\kappa_{\mu_2}+\kappa_{\mu_3})/2} s_{\mu_2^t}(q^\rho) \sum_\eta s_{\mu_1/\eta}(q^{\rho+\mu_2^t}) s_{\mu_3^t/\eta}(q^{\rho+\mu_2}), \tag{C.7}$$

where $s_{\mu/\nu}$ is the skew Schur function defined by

$$s_{\mu/\nu} = \sum_\eta c^\mu_{\nu\eta} s_\eta. \tag{C.8}$$

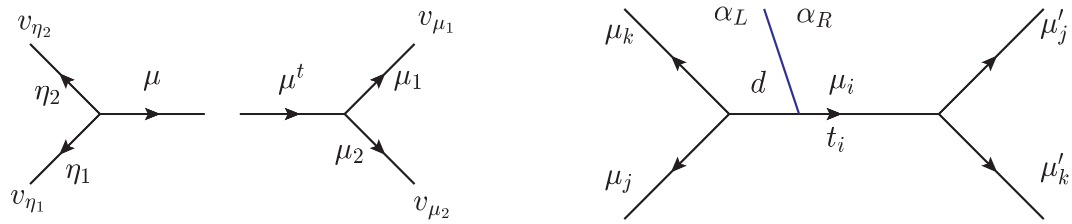

Figure 8: Gluing rule for topological vertex

We denote the Littlewood-Richardson coefficient by $c^{\mu}_{\nu\eta}$. The topological vertex enjoys the cyclic symmetry

$$C_{\mu_1\mu_2\mu_3}(q) = C_{\mu_3\mu_1\mu_2}(q) = C_{\mu_2\mu_3\mu_1}(q). \tag{C.9}$$

If some of μ_i's are the trivial representation $\emptyset$, the topological vertex simplifies as follows:

$$C_{\mu\emptyset\emptyset} = s_\mu(q^\rho), \tag{C.10}$$

$$C_{\mu\nu\emptyset} = q^{\kappa_\nu/2} s_\mu(q^\rho) s_{\nu^t}(q^{\rho+\mu}) = s_\nu(q^\rho) s_\mu(q^{\rho+\nu^t}). \tag{C.11}$$

The gluing rule for the topological vertex is

$$\sum_\mu C_{\mu\eta_1\eta_2}(-Q)^{|\mu|}(-1)^{n|\mu|} q^{-n\kappa_\mu/2} C_{\mu^t\mu_1\mu_2}, \tag{C.12}$$

where the integer n is defined by the exterior product of the vectors v_{μ_1} and v_{η_1}

$$n = v_{\mu_1} \wedge v_{\eta_1} = \det \begin{pmatrix} v^1_{\mu_1} & v^2_{\mu_1} \\ v^1_{\eta_1} & v^2_{\eta_1} \end{pmatrix}. \tag{C.13}$$

The vectors $v_{\mu_i} := (v^1_{\mu_1}, v^2_{\mu_1})$ and $v_{\eta_i} := (v^1_{\eta_1}, v^2_{\eta_1})$ are the directions of the corresponding legs in the toric diagram. In particular for inner branes, the gluing rule is generalized as follows:

$$\sum_{\mu_i, \alpha^L, \alpha^R} C_{\mu_j\mu_k(\mu_i\otimes\alpha^L)}(-1)^{s(i)} q^{f(i)} e^{-L(i)} C_{(\mu_i^t\otimes\alpha^R)\mu'_j\mu'_k} \ \mathrm{Tr}_{\alpha^L} V \ \mathrm{Tr}_{\alpha^R} V^{-1}, \tag{C.14}$$

with

$$L(i) = |\mu_i| t_i + |\alpha^L| r + |\alpha^R|(t_i - r), \tag{C.15}$$

$$f(i) = p\kappa_{\mu_i\otimes\alpha^L}/2 + (n+p)\kappa_{\mu_i^t\otimes\alpha^R}/2, \tag{C.16}$$

$$s(i) = |\mu_i| + p_i|\mu_i \otimes \alpha^L| + (n+p)|\mu_i^t \otimes \alpha^R|, \tag{C.17}$$

where $|\alpha\otimes\beta| = |\alpha| + |\beta|$ and $\kappa_{\alpha\otimes\beta} = \kappa_\alpha + \kappa_\beta$.

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